A Generalization of the Second Incompleteness Theorem and Some Exceptions to It

Dan E. Willard *


Abstract

This paper will introduce the notion of a naming convention and use this paradigm to both develop a new version of the Second Incompleteness Theorem and to describe when an axiom system can partially evade the Second Incompleteness Theorem.

1 Introduction

The Second Incompleteness Theorem states that no consistent axiom system is able to verify its own consistency when it attains a sufficient level of strength. Our objective will be to explore the generality and boundary-case exceptions for this effect under a class of axiom systems that fail to recognize successor as a total function and instead treat addition and multiplication as 3-way relations. (These formalisms cannot prove that $\forall x \exists y \ x + 1 = y$.) Instead, they will recognize the existence of an infinite collection of integers $0, 1, 2, ...$ by using an infinite number of constant symbols $C_0, C_1, C_2, ...$.

We will use the term "Naming Convention" to refer to a particular scheme for assigning integer values to named constant symbols. It will turn out that our ability to either generalize the Second Incompleteness Theorem or to find boundary-case exceptions to it will depend on the choice of naming method.

For simplicity, all our naming conventions will assign the values of 0, 1 and 2 to the first three constant symbols of $C_0, C_1$ and $C_2$. For $i \geq 3$, our Incremental, Additive and Multiplicative Naming Conventions will use the three different identities, specified by

*University of Albany, Email=dew@cs.albany.edu. Partially supported by NSF Grant CCR 99-02726
Equations (1)–(3), to define $C_i$’s value recursively in terms of $C_{i-1}$’s value. A more formal alternate definition of these naming conventions, that does not use the technically impermissible addition and multiplication function symbols, will appear in Section 3.

\begin{align*}
C_i &= C_{i-1} + 1 \\
C_i &= C_{i-1} + C_{i-1} \\
C_i &= C_{i-1} \ast C_{i-1}
\end{align*}

Integers having no formal names will be constructed from “named integers” via the operations of subtraction and division. (Thus since $5 = 8 - 2 - 1$, the term $C_4 - C_2 - C_1$ will define 5 under Equation (2)’s “additive” convention.)

One of our theorems will state that the additive naming convention does allow for some types of unusual boundary-type exceptions for the Hilbert-styled version of the Second Incompleteness Theorem. (A similar result will also hold for the incremental naming convention by a degenerate and thus somewhat weaker form of Section 4’s overall formalism.) On the other hand, Theorem 4 will state that every non-trivial, consistent axiom system $\alpha$, using Equation (3)’s multiplicative naming convention, is unable to recognize its own Hilbert consistency. (The result of Theorem 4 differs from our prior papers [43, 45, 46, 47, 48, 49, 50, 51] by examining a Hilbert mode of deduction, rather than the cut-free semantic tableaux deductive method.)

Some added notation is needed to describe our new results more formally. An axiom system $\alpha$ will be henceforth called Self-Justifying when:

i) one of $\alpha$’s theorems will assert $\alpha$’s consistency (using some reasonable definition of consistency), and

ii) the axiom system $\alpha$ is in fact consistent.

It is well known [13, 15, 29] that Kleene’s Fixed Point Theorem implies every r.e. axiom system $\alpha$ can be easily extended into a broader system $\alpha^*$ which satisfies condition (i). Kleene’s proposal [15] was essentially for $\alpha^*$ to contain all $\alpha$’s axioms plus the following additional axiom sentence:

\# There is no proof of 0=1 from the union of $\alpha$ with “THIS SENTENCE”.

2
Kleene noted that it was easy to apply the Fixed Point Theorem to formally encode a self-referencing statement, similar to the sentence #. The catch is that α* can be inconsistent even while its added axiom # formally asserts α*’s consistency. For this reason, Kleene, Rogers and Jeroslow [13, 15, 29] each emphatically warned their readers that most axiom systems similar to α* were useless on account of their inconsistency, although they were technically well-defined. This problem arises in both Gödel’s paradigm (where α extends Peano Arithmetic), as well as in more general settings [1, 3, 5, 11, 23, 28, 25, 32, 42, 47].

Our prior research [43, 46, 51] has developed several examples of self-justifying axiom systems using analogs of the axiom-sentence #, despite these limitations, mostly for the case where an axiom system recognized addition as a total function and treated multiplication as a 3-way relation that was not provably total. (Thus if $M(x, y, z)$ is a 3-way relation indicating that $x * y = z$ then our axiom systems $\alpha$ were unable to prove $\forall x \forall y \exists z M(x, y, z)$). In this context, we illustrated in [43, 46, 51] how several forms of the Kleene-like self-reflecting axiom # could construct self-justifying axiom systems under various definitions of semantic tableaux consistency.

The challenge in these articles was to assure the resulting axiom system $\alpha^*$ did not violate Part-ii of the definition of self-justification (by being inconsistent — due to a Gödel-like diagonalization argument — thereby making $\alpha^*$ useless albeit well-defined — and thus irrelevant). Our articles [43, 46, 51] showed this difficulty did not plague systems recognizing solely addition as a total function for some natural definitions of semantic tableaux consistency. On the other hand, [47] demonstrated the unavailability of this paradigm for circumventing the semantic tableaux version of the Second Incompleteness when an axiom system recognized multiplication as a total function.

Our goal in the present paper is to explore to what extent the preceding results about semantic tableaux definitions of consistency will generalize for definitions of consistency under Hilbert-style proof systems. Our research was stimulated in part by some theorems of Nelson, Pudlák, Solovay and Wilkie-Paris [21, 25, 32, 42]. In particular, a formula $\varphi(x)$ is called a **Definable Cut for** $\alpha$ iff $\alpha$ can prove the theorem:

$$\begin{align*}
\varphi(0) & \text{ AND } \forall x \varphi(x) \Rightarrow \varphi(x + 1) & \text{ AND } \forall x \forall y < x \varphi(x) \Rightarrow \varphi(y)
\end{align*}$$

Also, let $Q$ denote the well-known axiom system of Tarski-Mostowski-Robinson [36] that recognizes addition and multiplication as total functions, but contains little information about addition and multiplication beyond that. Tarski-Mostowski-Robinson [36], showed a general-
ization of the Gödel [9] and Rosser [28] versions of the First Incompleteness Theorem, which they called “Essential Undecidability”, was valid for $Q$, all its extensions, but none of its subsets. There have been several examples of generalizations of the Second Incompleteness Theorem for $Q$ beginning with Bezboruah-Shepherdson’s initial observations [5] on this subject (demonstrating that there were at least some particularized encodings of the provability predicate under which $Q$ was unable to prove its own Hilbert Consistency). Wilkie-Paris announced several extensions of the Second Incompleteness Theorem in [42] — demonstrating for instance that axiom systems as powerful as $IΣ_0 + Exp$ are unable to prove the Hilbert-consistency of formalisms as weak as $Q$. Pudlák [25] proved the following new form of the Second Incompleteness result:

**Theorem 1** (Pudlák 1985). *Suppose the consistent axiom system $α$ satisfies $α ⊃ Q$, $ϕ(x)$ denotes any of Equation (4)’s definable cuts, and Contra$α_p$ denotes any type of Gődel encoded statement asserting that $p$ represents a Hilbert-style proof of $0=1$ from $α$. Then $α$ can neither prove a conventional statement recognizing its own Hilbert consistency nor any weaker theorem of the form: “$∀p \; ϕ(p) \Rightarrow ¬Contra_α(p)$”.*

Let us say an axiom system $β$ **Canonically Formalizes Arithmetic** iff there exists two predicates $A(x, y, z)$ and $M(x, y, z)$ where $β$ proves:

i. $A(x, y, z)$ is true when $x + y = z$

ii. $M(x, y, z)$ is true when $x * y = z$

During several telephone calls in April of 1994, Robert Solovay communicated to us how the Theorem 1 (by Pudlák) could be combined with additional methods due to Nelson and Wilkie-Paris [21, 42], to obtain the following result about canonical formalizations of arithmetic:

**Theorem 2** (Solovay’s 1994 Modification [32] of Pudlák’s 1985 Version of the Second Incompleteness Theorem [25] using some of the methods of Nelson and Wilkie-Paris [21, 42]): *Let $β$ denote any consistent axiom system, canonically formalizing arithmetic, such that:

A) $β$ is able to prove Eq. (5)’s statement that successor is a total function.

$$∀x \; ∃z \; A(x, 1, z) \quad (5)$$
B) $\beta$ is able to prove a $\Pi_1$ type formatted theorem indicating the $A(x, y, z)$ and $M(x, y, z)$ predicates satisfy the associative, commutative, distributive and identity-element axioms for addition and multiplication.

Then $\beta$ must be unable to prove the non-existence of a Hilbert-style proof of $0=1$ from $\beta$ ’s set of axioms.

The following two observations will help explain the relationship between the preceding two theorems of Pudlák and Solovay:

1. Let us say that an axiom system $\beta$ recognizes addition and multiplication as total functions iff it can prove:

$$\forall x \forall y \exists z \ A(x, y, z) \quad \text{AND} \quad \forall x \forall y \exists z \ M(x, y, z) \quad (6)$$

The Theorem 2 by Solovay is an immediate consequence of the 1985 Theorem 1 by Pudlák when Equation (6) is made to replace Equation (5)’s formalism in Theorem 2’s hypothesis.

2. The reason that Solovay had communicated in April of 1994 to us the content of Theorem 2 is that we published in 1993 a paper [43] which showed that boundary-case exceptions to the semantic tableaux version of the Second Incompleteness Theorem do exist among axiom systems that recognize addition but not multiplication as a total function. Our paper [43] thus raised the question whether such boundary-case exceptions to the Second Incompleteness Theorem can be extended to Hilbert styled proofs under axiom systems that recognize solely addition as total? The Theorem 2 by Solovay is significant because it shows that no such extension is feasible for the Hilbert-style method of deduction.

A more detailed description of the formalisms that were employed by Theorems 1 and 2 and the related literature will appear during Section 2’s literature survey. At a first glance, it would certainly appear that Theorem 2 implies no meaningful axiom system can recognize its own Hilbert consistency — simply because Equation (5)’s axiom, recognizing successor as total, is the main threshold for activating Theorem 2 ’s formalism.

However, it turns out that Theorems 1 and 2 do allow for a subtle form of exception to exist for the Hilbert-styled version of the Second Incompleteness Theorem. This is because Equations (2) and (3)’s naming conventions have different and contrasting characteristics with
regards to the Second Incompleteness Theorem (that do not fall under the scope of either of these two theorems). In particular, one can construct axiom systems \( \alpha \), not recognizing successor as a total function and employing Equation (2)’s additive naming convention, which somewhat surprisingly can recognize their own Hilbert consistency. On the other hand, our Theorem 4 will show no analog of this boundary-type exception to the Second Incompleteness Theorem is available when a system employs Equation (3)’s multiplicative naming convention. (The point is that neither of these results are predicted by the earlier literature because the prior Theorems 1 and 2 presumed their axiom systems would recognize at least successor as a total function.)

Some added notation will help formalize our new results. Say a function \( F \) satisfies a **Non-Growth** property iff \( F(a_1, a_2, ...a_j) \leq \text{Maximum}(2, a_1, a_2, ...a_j) \) for all values of \( a_1, a_2, ...a_j \).

Seven examples of non-growth functions are *Integer Subtraction* (where \( x - y \) is defined to equal 0 when \( x < y \)), *Division-with-rounding* (where \( x \div y \) is defined to equal \( x \) when \( y = 0 \), and it equals \( \lfloor \frac{x}{y} \rfloor \) otherwise), *Maximum\((x, y)\)*, *Logarithm\((x) = 1 + \lfloor \log_2 x \rfloor\)*, *Predecessor\((x) = \text{Max}(x - 1, 0)\)*, *Root\((x, y) = \lceil x^{1/y} \rceil\)* and *Count\((x, j)\)* designating the number of “1” bits among \( x \)’s rightmost \( j \) bits. These functions are called the **Grounding Functions**.

We will use the term \( \Pi_1^- \) sentence to refer to a mathematical sentence identical to conventional Logic’s \( \Pi_1 \) sentence, except that the addition and multiplication function primitives are replaced by the grounding functions under the definition of a \( \Pi_1^- \) sentence. More formally, a Bounded Quantifier is defined as a phrase similar to either “\( \forall v \leq t \ \Psi(v) \)” or “\( \exists v \leq t \ \Psi(v) \)” where \( t \) is a term built out of the grounding function primitives. Let us call a formula \( \Delta^0^- \) when all its quantifiers are bounded, its two relation symbols are “\( = \)” and “\( \leq \)” and it uses the grounding function symbols. Then \( \Upsilon \) will be called \( \Pi_1^- \) when it is written in the form \( \forall v_1 \forall v_2 \ldots \forall v_n \ \Phi \), where \( \Phi \) is \( \Delta^0^- \). Using this terminology, our two main theorems are listed below:

**Theorem 3**. *Let \( A \) denote an arbitrary, consistent axiom system (using the grounding function language) all of whose \( \Pi_1^- \) theorems are logically valid under the standard model of the natural numbers. Suppose there also exists a \( \Delta^0^- \) encoding for the predicate HilbPrf\(_A\)(t, p) — which formally indicates that \( p \) is a Hilbert-style proof of the theorem \( t \) from axiom system \( A \). Then there exists a consistent axiom system, called ISCE(A) (which is formally defined in Section 4), whose constant symbols are defined via the additive naming convention’s*
methodology and which has an ability to

i. recognize the validity of all \( A \)’s \( \Pi_1^- \) theorems,

ii. recognize the assured non-existence of a Hilbert-style proof of 0=1 from its own set of proper axioms.

**Theorem 4.** No consistent axiom system \( \alpha \) is capable of proving a theorem affirming its own Hilbert-Consistency when it 1) contains all the multiplicative naming convention’s axioms, 2) retains an ability to prove all Peano Arithmetic’s \( \Pi_1^- \) theorems, and 3) satisfies a minor additional constraint, which the Definition 1 of Section 5 shall call the “Concise Encoding” property. (This “Concise Encoding” property is slightly stronger than a requirement that a \( \Delta^0_0 \) predicate identify all \( \alpha \)’s axioms.)

**Remark 1.** A stronger form of Theorem 4, called Theorem 4*, will receive an abbreviated proof in Appendix C. Theorem 4* will drop Theorem 4’s “Concise Encoding” assumption and also isolate a \( \Pi_1^- \) theorem \( W \) of Peano Arithmetic, where no consistent r.e. axiom system \( \alpha \supset W \), using the multiplicative naming convention, can formally verify its own Hilbert consistency.

Theorems 3, 4 and 4* are interesting partly because of their sharply contrasting features — where the simple replacement of the additive naming convention with a multiplicative convention is sufficient to activate the power of the Second Incompleteness Theorem.

A second interesting aspect requires some added notation to explain. Let us say a sequence of axioms \( S_1, S_2, S_3, ... \) defining the constant symbols \( C_1, C_2, C_3, ... \) is **Continuously Expanding** iff there exists a sequence of constants \( K_1, K_2, K_3, ... \) with \( K_i < K_{i+1} \) such that the set of axioms with Gödel numbers less than \( K_i \) is sufficient to generate a proof of the existence of an integer larger than \( K_{i+1} \). For example, the additive and multiplicative conventions, defined by Equations (2) and (3), satisfy the continuous expansion property — under the normalized assumption that the axiom defining \( C_i \) has a \( O(\text{Log}(i+2)) \) bit-length. On the other hand, Equation (1)’s incremental naming convention is not continuously expanding because it grows too slowly.

Ideally, an axiom system \( \alpha \) should satisfy the continuous expansion property — even if it fails to recognize successor as a total function — because the latter feature formalizes at least some type of weak notion of an infinite growth among integers. This explains the second
reason for our interest in Theorems 3 and 4. A degenerate version of Theorem 3 — with the incremental naming convention replacing the additive method — was established by the Theorem 3.4 of [46]. However Theorem 3.4’s axiom systems failed to satisfy the continuous expansion property because they relied upon the slow growing incremental naming convention. The additive convention will thus represent a useful compromise between the faster-growing multiplicative convention and the slower incremental convention — whose growth rate is simultaneously sufficiently slow to satisfy Theorem 3’s self-justification property while also sufficiently fast to satisfy the continuous expansion property. This distinction is significant because the terms $K_1, K_2, K_3, \ldots$ in the additive convention’s continuous expansion sequence will grow at a fast super-exponential rate.

One further definition is needed to describe a third theme in this article. Let $\text{Pred}^N(x)$ denote a compound operation that consists of $N$ iterations of the predecessor function (thereby causing $\text{Pred}^N(x) = x - N$). Say an axiom system $\alpha$ is **Infinitely Far-Reaching** iff there exists a finite subset of axioms $S \subset \alpha$ such that for an arbitrary integer $N$, the finite set $S$ is sufficient to prove $\exists x \text{Pred}^N(x) = 1$. Section 6 will outline how it is theoretically possible to construct an axiom system, which is simultaneously 1) Infintely Far Reaching, 2) able to verify its own Hilbert consistency, and 3) able to to prove all Peano Arithmetic’s $\Pi^1_1$ theorems.

Section 6’s axiom system will be highly awkward in its internal structure. However, it is noteworthy because one’s first intuition would be that the combination of the incompleteness effects described by Theorems 1, 2 and 4 would preclude a formalism from satisfying conditions 1–3 simultaneously.

**Over-all Perspectives and Objectives of this Article:** Any axiom system which fails to recognize successor as a total function is certainly a mixed blessing, whose weaknesses certainly cannot be overlooked or ignored. However, Gödel’s Incompleteness Theorem beckons one to wonder whether there are any boundary-case circumstances where the force of the Second Incompleteness Theorem can at least be partially evaded. Theorem 3 and Section 6’s ISINF(A) formalism will illustrate two types of partially curious forms of self-justifying systems that can be formally constructed. On the other hand, the prior literature’s Theorems 1 and 2 and our new Theorems 4 and 7 will indicate that there is a firm limit as to how far one can strengthen these boundary-case exceptions for the Second Incompleteness Theorem.

This topic has a different slant than the study of semantic tableaux in [43, 45, 46, 47, 48,
49, 50, 51] because the Theorem 2 (by Solovay) shows Hilbert-style exceptions to the Second Incompleteness Theorem cannot recognize even successor as a total function. It thus has a different flavor than the semantic tableaux formalisms of [43, 46, 48, 49, 50, 51], which recognized addition as a total function.

2 Background Literature

Let $S(x)$ denote the “successor” operation that maps the integer $x$ onto $x + 1$. A formula $\varphi(x)$ is called [11] a **Definable Cut** for an axiom system $\alpha$ iff $\alpha$ can prove:

$$\varphi(0) \text{ AND } \forall x \, \varphi(x) \Rightarrow \varphi[S(x)] \text{ AND } \forall x \forall y < x \, \varphi(x) \Rightarrow \varphi(y) \tag{7}$$

The articles [1, 3, 7, 8, 10, 11, 14, 16, 17, 21, 23, 24, 25, 27, 33, 38, 39, 41, 42] have studied Definable cuts. This concept is unrelated to a Gentzen-like “deductive cut rule”.

In a context where $\Upsilon(x)$ and $\varphi(x)$ denote two definable cuts, the formula $\varphi(x)$ has been called a **Thinning** of $\Upsilon(x)$ (relative to $\alpha$) iff $\alpha$ can prove:

$$\forall x \, \varphi(x) \Rightarrow \Upsilon(x) \tag{8}$$

The symbol $\lceil \Psi \rceil$ will henceforth denote $\Psi$’s Gödel number, and $\text{Prf}_{\alpha,D}(t,p)$ will denote that $p$ is a proof of the theorem $t$ from the axiom system $\alpha$ using the deduction method $D$. The system $\alpha$ will be said to recognize its **Cut-Localized D-consistency under** $\varphi(x)$ iff $\alpha$ can prove:

$$\forall p \{ \varphi(p) \Rightarrow \neg \text{Prf}_{\alpha,D}(\lceil 0 = 1 \rceil, p) \} \tag{9}$$

The recent literature has sought to identify exactly what triples $(\varphi, D, \alpha)$ have the property that $\alpha$ can prove (9)’s self-reflecting statement. Both positive and negative results, about the feasibility of constructing such triples have been established. Below is a summary of some of the recent results:

1. While GB Set Theory can prove its global Hilbert consistency is equivalent to the global Hilbert consistency of ZF-Set Theory, Hájek, Švejdar and Vopěnka [33, 41] have shown GB Set Theory cannot verify this equivalence within the range of every definable cut. In particular, Vopěnka-Hájek [41] identified a definable cut $\varphi$ where GB Set Theory can prove the validity of Equation (9) when $D$ denotes Hilbert deduction and $\alpha$ denotes ZF-set Theory. Moreover, Švejdar [33] has discussed some important generalizations of this
phenomena with regards to interpretability. (These two results are surprising because Pudlák [25] has proved that GB-Set Theory can never prove its Hilbert consistency on any of Equation (9)’s definable cuts. Thus, GB-Theory will view its Hilbert consistency as equivalent to ZF’s Hilbert consistency in a global sense — but not in a sense localized inside each definable cut.)

2. In a context where $D$ denotes a second order generalization of Gentzen’s sequent calculus and $\alpha$ is a formalization of Arithmetic among the natural numbers, Kreisel-Takeuti [19] identified a definable cut where their system could prove (9)’s statement of about its D-consistency.

3. Nelson [21] demonstrated that a variant of the Tarski-Mostowski-Robinson axiom system $Q$ (with Linear Ordering) can corroborate Equation (9)’s statement about its Cut-Localized consistency when $D$ denotes Herbrand deduction and the cut $\varphi$ is carefully chosen.

4. Pudlák proved two results about Equation (9) in [25]. The first was a substantial generalization of Items 2 and 3 (above), which showed that every “sequential” [11] axiom system of finite cardinality can be associated with a definable cut $\varphi$ such that $\alpha$ can prove Equation (9) is valid for $\varphi$ when $D$ denotes any cut-free method of deduction. The Theorem 1 , mentioned earlier in Section 1, also appeared in Pudlák’s article [25]. It indicated that no consistent $\alpha \supset Q$ can prove any form of Equation (9)’s sentence about itself when $D$ denotes Hilbert deduction.

5. Wilkie-Paris [42] used the theory of definable cuts to prove several new versions of the Second Incompleteness Theorem in [42]. One of their most surprising results was the demonstration that an axiom system as strong as $\Sigma_0 + Exp$ is unable to prove the Hilbert consistency of $Q$.

6. Let $\text{Log}^K(z)$ denote the operation $\text{Log}((\text{Log}(\text{Log}(\text{Log}(z))))$ — where $K$ is an integer indicating the number of iterations of Log here. Several articles [11, 17, 21, 23, 25, 27, 39, 42] have credited some unpublished comments of Robert Solovay for observing that for each integer $K$ and definable cut $\Upsilon(x)$, there exists a thinning $\varphi(x)$ of $\Upsilon(x)$ where most (even weak) arithmetic axiom systems $\alpha$ can prove:
∀ p [ ϕ(p) ⇒ ∃ q  Υ(q) ∧ Log^K(q) = p ] AND

∀ p [ ϕ(p) ⇒ ∃ r  ϕ(r) ∧ Log^K(r) = 2 · Log^K p ] (10)

This never-published but often cited result by Solovay has been gainfully employed by a large part of the literature on Definable Cuts. (A very nice proof of it appears on pages 172–173 of the Hájek-Pudlák textbook [11]).

7. As noted in Section 1, Robert Solovay privately communicated to us in April of 1994 that he knew how to combine the formalisms of Nelson, Pudlák and Wilkie-Paris [21, 25, 42] to prove the Theorem 2, which can refine the 1985 Pudlák-like Incompleteness Effect — so it will apply to axiom systems recognizing merely successor as a total function. To help publicize this never-published result, we published a proof of a weak version of Theorem 2 — with attribution to our communications with Solovay — in Appendix A of our year-2001 article [46]. This appendix’s result is not quite as strong as the broader version of Solovay’s Theorem 2, but it has the virtue of having a pleasantly short 4-page proof.

8. Paris-Dimitracopoulos [23] (and Pudlák [25] using a different method) both observed that for an arbitrary initial definable cut Υ, it is not always automatically feasible to construct a thinner cut ϕ that is closed under the operation of Exponentiation.

9. Characterizations of relative interpretability for finitely axiomatized sequential theories were independently developed by Friedman and Pudlák in [8, 25]. See some papers by Smoryński and Visser [31, 37, 40] for some very detailed descriptions of these contributions by Friedman and Pudlák.

10. Let Con^D(ΙΣ₀) denote the variant of Equation (9) where 1) α =ΙΣ₀, 2) D designates the Hilbert deduction method and 3) ϕ represents Equation (9)’s employed definable cut. Krajíček [16] proved that for any ΙΣ₀ cut Υ there exists a thinner ΙΣ₀ cut ϕ such that the theorem ΙΣ₀ + Con^D(ΙΣ₀) + ¬Con^Υ(ΙΣ₀) is consistent, and that an analog of this construct holds for any finitely generated sequential theory under the Wilkie-Paris notion of a restricted proof [42]. Visser [39] generalized this construct to show that many consistent axiom systems, such as Q, ACA₀ and GB-Set Theory, have the property that no finite consistent extension of themselves implies there exists Hilbert-style proofs of 0=1 from themselves simultaneously positioned in each of their definable cuts.
11. Buss, Ignjatovic, Krajíček, Pudlák and Takeuti [6, 7, 18, 26, 35] have explored the problem of separating Buss’s axiom system \( S_2 \) into its sub-systems called \( S_1^2, S_2^2, S_3^2, \ldots \). This problem is connected to a variety of open questions about NP, the Polynomial Hierarchy and Cook’s system PV. Several of these articles have used the properties of definable cuts in the course of their analysis. For example using the theory of definable cuts, Buss and Ignjatovic [7] showed that \( PV \) is unable to confirm the consistency of its induction-free fragment, called \( PV^- \).

A small amount of additional notation is needed to describe one further development in proof theory. Let \( D \) denote any deduction method such as Herbrand deduction, Hilbert deduction, semantic tableaux etc. Let \( \tilde{\text{O}}_D(\alpha) \) denote the classic Gödel Diagonalization Sentence that states:

* There is no proof of this sentence from axiom system \( \alpha \) using deduction method \( D \).

Let \( \text{Log}^\lambda(z) \) denote the operation \( \text{Log}(\text{Log}(...(\text{Log}(z)))) \) — where \( \lambda \) is an integer indicating the number of iterations of \( \text{Log} \) here. Let \( \text{ShortPrf}^\lambda_{\alpha,D}(x,y,z) \) denote a \( \Delta_0^\text{c} \) formula indicating that \( y \) represents a proof of the theorem \( x \) from the axiom system \( \alpha \) using deduction method \( D \) and that \( y = \text{Log}^\lambda(z) \). Also, let \( \tilde{\text{O}}_D^\lambda(\alpha) \) denote the generalization of the \( \tilde{\text{O}}_D(\alpha) \) sentence that has the \( \text{ShortPrf}^\lambda_{\alpha,D}(x,y,z) \) construct replace conventional deduction. This construct, called a Generalized Gödel Sentence, is defined formally as:

** In a context where one employs the “\( \text{ShortPrf}^\lambda_{\alpha,D}(x,y,z) \)” notation, there exists no code \( (y,z) \) that “proves” this sentence.

Several articles have noticed how Generalized Gödel sentences can serve as a helpful intermediate step for developing a variety of generalizations and applications of the Incompleteness Theorem. This is essentially because Definable Cuts are not always ideally suited for generalizing the Second Incompleteness Theorem when \( D \) represents a cut-free deduction method, such as Herbrand deduction, semantic tableaux or the cut-free sequent calculus. Thus, Takeuti [35] used generalized Gödel sentences to analyze the 3-way relationship between the properties of NP, Buss’s Bounded Arithmetic and some generalizations of Gentzen’s sequent calculus. Generalized Gödel sentences have also been used to answer a 20-year old Paris-Wilkie open question [24] about whether any extension of \( I\Sigma_0 \) can prove its own cut-free consistency. In
this context, Adamowicz-Zbierski [1, 3] used Generalized Gödel sentences to show that a cut-free version of the Second Incompleteness Theorem was valid at the level of $\Pi_0^1 + \Omega_1$. Willard's two papers [45, 47] strengthened this result to show that such a cut-free second incompleteness effect applied to all extensions of $\Pi_0$ and most extensions of $Q$, and Salehi [30] has recently explored some other types of interesting proofs of this incompleteness effect.

One type of formalism that combines the theories of definable cuts and of Generalized Gödel Sentences into a hybridized framework can be found in [49]. It examines a hierarchy of several increasingly elaborate definitions of semantic tableaux consistency, where one wishes to determine at what level in this hierarchy does the semantic tableaux version of the Second Incompleteness Theorem become operative for axiom systems that recognize addition (but not multiplication) as a total function. Via a hybridizing of the theory of definable cuts with the notion of a Generalized Gödel Sentence, [49] obtained an incompleteness result for this hierarchy.

Finally, let us return to the Theorems 1 and 2 of Pudlák and Solovay. Theorem 2 was described by Solovay as resulting from the combined research efforts of himself and of Nelson, Pudlák and Wilkie-Paris. It stated that no axiom system can prove its own Hilbert consistency and also recognize successor as a total function. An important fact is that Theorem 2 does not apply to semantic tableaux deduction, since our papers [43, 46, 48, 51] had established that semantic tableaux boundary-case exceptions to the Second Incompleteness Theorem can recognize addition as a total function. Hence, there naturally arises the question to consider whether some type of axiom system, dropping the assumption that successor is a total function, can recognize its own Hilbert consistency? Our Theorems 3, 4, 6 and 7 will partially answer this question by providing some closely matching positive and negative results.

### 3 Main Notation Conventions

This section will offer a brief summary of the formal notation used in this paper. An intuitive (but not formal) description of the three incremental, additive and multiplicative naming conventions was provided by Equations (1)–(3) in the Introduction Section. These equations should not be regarded as the formal definitions of these three naming conventions because the addition and multiplication function symbols were employed in their recursive definitions. To rectify this problem, we need to rewrite these equations in a language that has the grounding function symbols of predecessor, subtraction and division-with-rounding replace the roles of
addition and multiplication.

Our first task is to define the constant symbols \( C_0 \), \( C_1 \) and \( C_2 \) to represent the integers of 0, 1 and 2. Since Predecessor(0) = 0, this is done below:

\[
\text{Pred}(C_0) = C_0 \land C_1 \neq C_0 \land \text{Pred}(C_1) = C_0 \land \text{Pred}(C_2) = C_1
\]  

(11)

Next, let \( \text{ADD}(x, y, z) \) denote “\( z \geq x \land z - x = y \)”, and \( \text{MULT}(x, y, z) \) denote \( [ (x = 0 \lor y = 0) \Rightarrow z = 0 ] \land [ (x \neq 0 \land y \neq 0) \Rightarrow (\frac{x}{y} = y \land \frac{y - 1}{x} < y) ] \).

These two \( \Delta^- \) predicates do not technically use the Multiplication and Addition function symbols, but a triple satisfies them only when respectively \( x + y = z \) and \( x \cdot y = z \) are true.

Finally, let \( j_1, j_2, j_3 \ldots, a_1, a_2, a_3 \ldots \) and \( b_1, b_2, b_3 \ldots \) denote the axiom-sequences, used by our incremental, additive and multiplicative naming conventions. The \( \text{ADD}(x, y, z) \) and \( \text{MULT}(x, y, z) \) predicates allow us to rewrite the description of these axioms (from Equations (1)–(3)) in a revised language that replaces the Addition and Multiplication function symbols with the operations of Predecessor, Subtraction and Division-with-rounding. Thus, the first formal axiom of these three conventions (i.e. \( j_1 \), \( a_1 \) and \( b_1 \)) is defined by Equation (11).

For \( i \geq 2 \) their additional axioms are:

\[
\begin{align*}
  j_i &= \text{df} \quad \text{ADD}(C_i, 1, C_{i+1}) \quad (12) \\
  a_i &= \text{df} \quad \text{ADD}(C_i, C_i, C_{i+1}) \quad (13) \\
  b_i &= \text{df} \quad \text{MULT}(C_i, C_i, C_{i+1}) \quad (14)
\end{align*}
\]

Let \( \text{Bit}(x, i) \) denote the \( i \)-th rightmost bit associated with the binary encoding of the integer \( x \). Equation (15) shows how one can encode the term \( \text{Bit}(x, i) \) in our grounding function language as a compound function built out of the grounding operations of count and subtraction:

\[
\text{Bit}(x, i) = \text{Count}(x, i) - \text{Count}(x, i - 1)
\]  

(15)

Let \( \text{Pred}^N(x) \) denote an operation that consists of \( N \) iterations of the predecessor function (thereby causing \( \text{Pred}^N(x) = x - N \)). Section 1 had defined our grounding function \( \text{Log}(x) \) to formally represent the integer value of \( 1 + \lceil \text{Log}_2 x \rceil \). Let \( L_d \) denote an integer constant that represents the bit-length of the integer \( d \)’s binary encoding, and let its binary representation be formally encoded by the bit-sequence \( \beta_1, \beta_2 \ldots \beta_{L_d} \). In this context, let \( \sigma_d(x) \) denote Equation (16)’s \( \Delta^- \) formula. It has the characteristic that the only integer \( x \) satisfying (16) is the integer \( d \) itself.
Bit(x, Log(x)) = β_1 \land Bit(x, Pred(Log(x))) = β_2 \land \ldots \land Bit(x, Pred^{\omega-1}(Log(x))) = β_L \land Pred^{\omega-1}(Log(x))) = 1 \quad (16)

Since \lceil \Phi \rceil denotes \Phi’s Gödel number, our notation convention will imply that the only integer x satisfying σ_{\lceil \Phi \rceil}(x) is \Phi’s Gödel number.

Our method for encoding Gödel numbers is defined in Appendix A. This convention is ideally compact insofar as its encoding of a particular sentence or proof will have a bit-length that is essentially proportional to the effort to write down such an object by hand. Other examples of ideally-compact Gödel encoding schemes have been described by for example Hájek-Pudlák and Wilkie-Paris in [11, 42]. It is therefore probably unnecessary for a reader to examine Appendix A in much detail. Essentially, Appendix A can be omitted provided the reader keeps in mind that the bit-length needed for encoding the name of the symbol C_i, as well as for encoding the accompanying axioms j_i, a_i or b_i, will have an ideally compressed O(Log(i + 1)) order of magnitude.

Our results also generalize in various forms for non-compressed encodings, where C_i has an O(i) bit-length instead. We omit discussing these generalizations here because uncompressed encodings are inherently unnatural.

4 The ISCE Formalism and its Generalizations

This section will be devoted to proving Theorem 3 and defining its accompanying ISCE(A) formalism. In our discussion, a Π_1^- sentence will be called 2-reduced iff it uses only the constant symbols for the three natural numbers 0, 1 and 2. This restriction is not serious because every unreduced Π_1^- sentence has a 2-reduced counterpart that is equivalent to it under Equation (16)’s σ—formalism. For instance, if φ(a, b) is a 2-reduced Δ_0^- formula and if \tilde{k} is an arbitrary integer, then the unreduced Π_1^- sentence “∀x φ(x, \tilde{k})” is equivalent under the standard model to the 2-reduced Π_1^- sentence below:

\forall x \forall y \{ \sigma_k(y) \Rightarrow \phi(x, y) \} \quad (17)

A similar transformation is obviously available to map every unreduced Π_1^- sentence onto its equivalent 2-reduced counterpart. Moreover in the context of axiom systems that recognize the existence of an unending sequence of natural numbers 0, 1, 2, ..., by using for example the additive naming convention, the 2-reduced sentence (17) is provably equivalent to its
unreduced counterpart “∀x φ(x, k)” . Thus, there is no difficulty when we employ 2-reduced Π₁⁻ sentences in this section. Often we will omit the phrase “2-reduced” when it is evident that the Π₁⁻ sentence is 2-reduced.

The acronym “ISCE” will stand for *Introspective Semantics with Continuous Expansion*. It will be defined similarly to [43, 46]’s IS(A) and ISREF(A) axiom systems except that ISCE(A) will hybridize Section 1’s continuous expansion property with an ability of a self-reflecting formalism to recognize its own Hilbert consistency. Given an initial axiom system A , ISCE(A) will thus be defined to be a self-justifying axiom system, capable of proving all of A’s Π₁⁻ theorems, and consisting of the following four axiom groups:

**GROUP-ZERO:** This axiom group will consist of the axiom sentences a₁, a₂, a₃ ... used by Equations (11) and (13) of Section 3’s additive naming method. It differs from the Group-Zero axioms in our earlier papers by using an additive (rather than incremental) naming method. (Note that additive naming is continuously expanding, but incremental naming is not.)

**GROUP-1:** This group will consist of a finite set of Π⁻₁ axioms defining ISCE’s grounding functions. This means that for each grounding function G and set of numbers k, k₁, k₂,...kₘ, the combination of the Group-Zero and Group-1 axioms will imply \( G(k₁, k₂,...kₘ) = k \) when this sentence is true. The Group-1 scheme will also assign the “=” and “<” predicates their usual logical properties. Any finite set of 2-reduced Π⁻₁ sentences that meet the preceding conditions is adequate. Table I of [46] provides one example of a suitable set of Group-1 axioms.

**GROUP-2:** Let \([Σ]\) denote Φ’s Gödel number, and HilbPrfₐ(x, y) denote a \(Δ₀⁻\) formula indicating y is a proof from axiom system A of the theorem x. Suppose that A uses the same grounding function symbols as ISCE(A), and it therefore generates a set of Π₁⁻ theorems. For each Π₁⁻ sentence Φ, our prior papers had the Group-2 schema contain an axiom of the form:

\[
∀ y \{ \text{HilbPrf}_A ( [Φ], y ) \Rightarrow Φ \}
\]  

The Group-2 axioms of ISCE(A) will have essentially the same format, except we cannot use exactly Equation (18)”s formal equation because ISCE’s language does not contain
constant symbols for every natural number. It is easy to overcome this difficulty by using Equation (17)’s formalism for mapping an unreduced $\Pi_1^-$ sentence onto its 2-reduced counterpart. Thus, the effective equivalent for (18)’s $\Pi_1^-$ sentence is:

$$\forall y \forall x \{ [ \sigma_{[\Phi]}(x) \land \{ \text{HilbPrf}_{\text{ISCE}(A)}(x,y) \} ] \Rightarrow \Phi \}$$  \hspace{1cm} (19)$$

ISCE(A) will contain one such axiom for each 2-reduced $\Pi_1^-$ sentence $\Phi$.

**GROUP-3:** ISCE(A)’s Group-3 axiom will consist of a single $\Pi_1^-$ sentence that essentially corresponds to the following statement:

$$\blacklozenge \quad \text{“There is no Hilbert-style proof of 0=1 from the union of the Group-0, 1 and 2 axioms with THIS SENTENCE (referring to itself)”}.$$  

We have already illustrated in several papers [44, 46] how similar Kleene-like self-referential $\Pi_1^-$ constructions to the sentence $\blacklozenge$ above were possible. In essence, the $\Pi_1^-$ encoding of $\blacklozenge$ rests on constructing a special $\Delta_0^-$ formula called “ HilbPrf ISCE(A) (x,y) ” such that Equation (20) (below) can be roughly thought of as being semantically equivalent to the sentence $\blacklozenge$. The exact details of how HilbPrf ISCE(A) (x,y) shall receive a $\Delta_0^-$ encoding will be explained in the course of Lemma 1’s proof.

$$\forall x \forall y \neg \{ \sigma_{[0=1]}(x) \land \text{HilbPrf ISCE(A)}(x,y) \}$$ \hspace{1cm} (20)$$

**Lemma 1.** There exists a $\Delta_0^-$ encoding for the HilbPrf ISCE(A) (x,y) predicate which will cause Equation (20)’s formal mathematical sentence to be semantically equivalent to the content of the Group-3 statement $\blacklozenge$.

**Proof Sketch.** Our justification of Lemma 1 will be somewhat abbreviated because analogous techniques were previously used in our paper [46] to define its Group-3 axiom. In particular, let us employ the following notation:

1. Subst (g, h) will denote Gödel’s classic substitution formula, which yields TRUE when h is a formula identical to g — except that it replaces all g’s free variables with an integer term equal to g’s Gödel number.

2. UNION(A) will denote the union of ISCE(A)’s Groups 0, 1 and 2 axioms.
3. \( \text{HilbPrf}_{\text{UNION}(A)}(t, p) \) is a formula stating that \( p \) represents a Hilbert-style proof of the theorem \( t \) from the axiom system \( \text{UNION}(A) \).

4. \( \text{ExtraPrf}_{\text{UNION}(A)}(t, h, p) \) is a formula stating that \( p \) represents a Hilbert-style proof of the theorem \( t \) from the union of the axiom system \( \text{UNION}(A) \) with the additional axiom-sentence whose Gödel number is represented by the integer \( h \).

5. \( \text{FixPointPrf}(t, g, p) \) will be an abbreviation for the sentence:

\[
\text{HilbPrf}_{\text{UNION}(A)}(t, p) \lor \\
\exists h \leq p \left[ \text{Subst}(g, h) \land \text{ExtraPrf}_{\text{UNION}(A)}(t, h, p) \right]
\]  

(Appendixes B through D of our article [46] had explained in meticulous detail how one could use the theory of LinH functions [11, 17, 52] to provide \( \Delta^0 \) encodings for each of the formulae of \( \text{Subst}(g, h) \), \( \text{HilbPrf}_{\text{UNION}(A)}(t, p) \) and \( \text{ExtraPrf}_{\text{UNION}(A)}(t, h, p) \). It therefore follows that Equation (21)'s compound formula of \( \text{FixPointPrf}(t, g, p) \) also has a \( \Delta^0 \) encoding. This in turn implies that Equation (22) (below) has a \( \Pi^1_1 \) encoding:

\[
\forall x \forall y \neg \{ \sigma_{[0=1]}(x) \land \text{FixPointPrf}(x, g, y) \}
\]  

The only free variable in Equation (21) is \( g \). Let \( \Psi(g) \) denote this formula, and \( \theta \) denote \( \Psi(g) \)'s Gödel number. Also, let \( C^* \) denote the particular constant employed by our additive naming convention that represents the least power of 2 greater than \( \theta \). Let \( \bar{\theta} \) denote a term whose value equals \( \theta \) — where \( \bar{\theta} \)'s encoding has a binary-like format with its term beginning with the constant symbol \( C^* \) and then subtracting the \( \log_2 \theta \) or fewer needed powers of 2 smaller than \( C^* \), so that \( \theta \)'s exact value is produced. The Equation (23)'s sentence can then be viewed as being the formal semantic representation of either Equation (20)'s sentence or the equivalent Group-3 statement \( \diamondsuit \).

\[
\forall x \forall y \neg \{ \sigma_{[0=1]}(x) \land \text{FixPointPrf}(x, \bar{\theta}, y) \}
\]  

Hence, this proof has confirmed Lemma 1's claim by showing how one can view (23)'s "FixPointPrf\( (x, \bar{\theta}, y) \)" predicate as being the \( \Delta^0 \) formalization of (20)'s "\( \text{HilbPrf}_{\text{ISCE}(A)}(x, y) \)" formula — thereby causing Equation (23) to capture the semantic meaning of the statement \( \diamondsuit \).  \( \square \).
Remark 2. The proof of Lemma 1 had assigned $\text{HilbPrf}_{\text{ISCE}(A)}(x, y)$ a $\Delta_0$ encoding because Lemma 1’s formalism would become substantially less significant if this requirement was relinquished.

The formal statement of Theorem 3 was given in Section 1. It was essentially that the axiom system ISCE$(A)$ will be automatically consistent whenever:

1. All $A$’s $\Pi_1^-$ theorems are logically valid under the standard model, and
2. The formula $\text{HilbPrf}_A(x, y)$ has a $\Delta_0$ encoding.

The proof of Theorem 3 is given below:

Proof. We will prove Theorem 3 by employing a proof-by-contradiction. If this theorem was false, then there would exist an axiom system $A$ satisfying conditions 1 and 2 (above) where ISCE$(A)$ is inconsistent. In this context, let $p$ represent the minimal sized proof of $0 = 1$ from ISCE$(A)$.

Let $G$ denote the Gödel number of Equation (20)’s Group-3 axiom-sentence. Let $k$ denote the least integer greater than $1 + \log_2(G)$ such that the additive naming convention’s particular axiom $a_{k+1}$ does not appear inside the proof $p$. Also, define the integer $j$ to equal $2^k$. Consider a model-theoretic interpretation, $M^k_j$, of the axioms appearing in $p$, that has the following two properties:

A) The interpretation $M^k_j$ will assume that $C_i = 2^{i-1}$ when $1 \leq i \leq k + 1$ and that $C_i = 0$ when either $i \geq k + 2$ or $i = 0$.

B) The interpretation $M^k_j$ will assume no integer larger than $j = 2^k$ exists. By this we mean that both the “named integers”, associated with explicitly denoted constant symbols $C_0, C_1, C_2, \ldots$, and also the “unnamed integers” (which are not assigned a designated constant symbol) are presumed under the interpretation $M^k_j$ to assume values no larger than $j = 2^k$.

Since the additive naming convention’s $i$–th axiom $a_i$ causes $C_{i+1} = 2C_i$, the model $M^k_j$ is obviously consistent with all this convention’s axioms $a_1, a_2, a_3, \ldots$ except for the axiom $a_{k+1}$, which plainly violates Requirement-A.

Also because the grounding functions are non-growth functions, each 2-reduced $\Pi_1^-$ sentence, which is valid under the standard model, must be automatically valid also in the finite model.
Thus, all the Group-1 axioms are valid in $M_j^k$. Also, all ISCE($A$)'s Group-2 axioms are valid in the finite model $M_j^k$ because of the prior page’s Assumption-1 combined with the fact that ISCE($A$)'s Group-2 axioms are $2$-reduced $\Pi_1^0$ sentences.

Let $Z$ denote the set of all Group 0, 1 and 2 axioms except that the axiom $a_{k+1}$ is excluded from the set $Z$. All the axioms in $Z$ were shown by the preceding two paragraphs to satisfy the model $M_j^k$. However since $p$ represents a proof of $0=1$, some axiom in $p$’s proof must be invalid in this model $M_j^k$. This implies that Equation (20)’s Group-3 axiom must be automatically invalid in the model $M_j^k$ (because $a_{k+1}$ does not appear in the proof $p$ and some axiom inside the proof $p$ must violate $M_j^k$).

The invalidity of (20)’s Group-3 axiom under the model $M_j^k$, in turn, implies there must exist a proof $q$ from ISCE($A$) of $0=1$ such that:

$$q \leq j = 2^k$$

(24)

We will now use Equation (24) to prove $q < p$, a result that will finish our proof-by-contradiction (by contradicting the initial assumption that $p$ was the minimal proof of $0=1$ from ISCE($A$) ). The preceding paragraph had demonstrated how $p$’s proof of $0=1$ had employed Equation (20)’s Group-3 axiom as one of its essential steps. Let $G$ again denote the Gödel number of this Group-3 axiom. Since Equation (23)’s Group-3 axiom is a necessary step in $p$’s proof, it is obvious that $p$’s bit-length is certainly more than three times the length of this Group-3 axiom (under all natural methods for the Gödel encoding of a proof, including the particular encoding methods we have sketched in Appendix A). Hence, Equation (25) formalizes this inequality:

$$\log_2(p) > 3 \log_2(G)$$

(25)

Moreover, let $L$ denote the number of Group-zero axioms appearing inside the proof $p$. Then the quantity $k$ (defined in the second paragraph of this proof) obviously satisfies the inequality:

$$k \leq L + 1 + \log_2(G)$$

(26)

Under all usual encoding conventions, including the particular convention sketched in Appendix A, the number of Group-zero axioms in $p$’s proof will be less than $\frac{1}{3} \log_2(p)$. (This is because every Group-zero axiom uses more than 3 bits of storage.) Equation (27) formalizes this inequality:

$$L < \frac{1}{3} \log_2(p)$$

(27)
In order to now finish our proof-by-contradiction, we must show that the combination of Equations (24) – (27) imply that:

\[ q < p \]  

(28)

The justification of (28)’s inequality is quite trivial. It is because the presence of the factors “3” in Equation (25) and \( \frac{1}{3} \) in (27) assure (via Equations (24) and (26)) that \( \log(q) \) is no more than essentially \( \frac{2}{3} \) of the size of \( \log(p) \) (whenever \( p \) has more than a tiny length of say roughly 100 bits).

Equation (28)’s inequality \( q < p \) clearly contradicts our initial assumption that \( p \) was the minimal proof of \( 0=1 \) from ISCE(A). Hence, our proof-by-contradiction has shown that Theorem 3 must be valid because otherwise the required minimal size of the proof \( p \) will be violated. \( \square \)

**Significance of Theorem 3.** Part of what makes Theorem 3 interesting is that there is a tight match between the theorem’s boundary-case exceptions to the Second Incompleteness Theorem and Theorem 4’s generalization for it (because these two theorems differ only by replacing the former’s additive naming convention with the latter’s multiplicative convention).

A second interesting aspect of ISCE(A) is that it satisfies a “continuous expansion property” (where the concerned axiom system can be associated with a sequence of integers \( K_1 < K_2 < K_3 < K_4 \ldots \), such that the union of all the axioms with Gödel number less than \( K_i \) can be combined to prove the existence of an integer larger than \( K_{i+1} \)). What makes this expansion paradigm especially interesting is that the sequence \( K_1, K_2, K_3 \ldots \) grows at a very fast rate as its index goes to infinity. In particular, let \( 2^i \) designate another way of writing the number \( 2^i \), and let \( 2^i_{m+1} \) denote the quantity \( 2^{2^m} \).

The sequence \( K_1, K_2, K_3 \ldots \) will actually grow faster than the elements in any sequence of the form \( 2^1_m, 2^2_m, 2^3_m, \ldots \), for any fixed \( m \). More precisely for a carefully chosen positive constant \( r < 1 \), ISCE(A)’s additive naming convention will have its continuous expansion sequence grow at an essentially super-exponential rate which satisfies (29)’s inequality:

\[ K_{i+1} > 2^{(K_i)^r} \]  

(29)

There is also a third interesting facet of the ISCE(A) axiom system. For a prenex normal sentence \( \Psi \), let \( \Psi^x \) denote a sentence identical to \( \Psi \) except every previously unbounded universally quantified variable in \( \Psi^x \) is bounded by \( x \). (All existential quantifiers and bounded universal quantifiers have their ranges unchanged under this definition.) Also let \( \text{TangPred}(x) \)
and \( \text{ TangDiv}(x) \) denote the following two formulae:

\[
\begin{align*}
\text{a. TangPred}(x) &= \{ \exists v \ x = v - 1 \} \\
\text{b. TangDiv}(x) &= \{ \exists v \ x < \frac{v}{2} \}
\end{align*}
\]

In a notation where \( \text{Tangible}(x) \) denotes either \( \text{TangPred}(x) \) or \( \text{TangDiv}(x) \), an axiom system \( \alpha \)'s **Tangibility Reflection Principle** for the sentence \( \Psi \) is defined to be the assertion:

\[
\forall x \ [ \exists y \ \text{HilbPrf}_\alpha(\ceil{\Psi}, y) \land \text{Tangible}(x) ] \Rightarrow \Psi^x \quad (30)
\]

Our prior papers \[44, 46\] had illustrated two examples of self-justifying systems, called \( \text{ISREF}(A) \) and \( \text{ISTR}(A) \), that could prove the correctness of their respective \( \text{TangPred} \) and \( \text{TangDiv} \) reflection principles for every prenex normal sentence \( \Psi \). These papers also explained why a system \( \alpha \)'s ability to prove the validity of its tangibility reflection principle for each sentence \( \Psi \) is a much stronger form of self-justifying assertion than \( \alpha \)'s mere ability to prove the non-existence of a proof of \( "0=1" \) from itself (see footnote 1).

The term “incremental naming convention” had not appeared in our article \[46\]. However, its Theorem 3.4 had implicitly presumed the presence of an incremental naming method, since its encoding convention had employed a constant symbol with an \( O(\log(i + 2)) \) bit-length for representing each integer \( i \). It turns out that one can readily generalize \[46\]'s formalism for the case where \( \alpha \) employs Equation (13)'s stronger additive naming convention instead of the weaker incremental method. Thus by hybridizing Theorem 3’s proof with our earlier formalisms from Section 3 of \[46\], one can obtain:

**Theorem 3**. Let us again use the assumptions from Theorem 3’s hypothesis that the axiom system \( A \) satisfies the following two conditions:

1. All \( A \)'s \( \Pi_1^- \) theorems are logically valid under the standard model, and

2. The formula \( \text{HilbPrf}_A(x, y) \) will have a \( \Delta_0^- \) encoding.

Then it is possible to devise a consistent axiom system \( \alpha \) that can simultaneously prove all \( A \)'s \( \Pi_1^- \) theorems, verify the correctness of its \( \text{TangPred} \) reflection principle and support the additive naming convention.

---

1. For instance, the Theorem 7.2 from \[46\] shows that the analog of Equation (30)'s reflection principle is typically infeasible when its \( \text{Tangible}(x) \) phrase is removed. Thus, Equation (30) is both more expressive than the mere statement that no proof of \( 0=1 \) exists, and it is close to the maximal type of self-justifying statement feasible.
In the interests of brevity, Theorem 3*'s proof shall not be presented here. This proof was omitted mostly because it is a natural hybrid of the Theorem 3’s proof with the added formalism that [46] used to prove its Theorem 3.4. A second reason for focusing on the weaker but simpler version of Theorem 3 is that the contrast between the opposing positive and negative results of Theorems 3 and 4 is easier to visualize under Theorem 3’s simpler version.

5 Incompleteness under the Multiplicative Naming Convention

This section will discuss the incompleteness properties of an axiom system that employs Equation (14)'s multiplicative naming convention.

**Definition 1.** Let us recall Equation (16) defined \( \sigma_n(x) \) to be a \( \Delta_0^- \) formula that was satisfied only when \( x \) corresponded to the natural number \( n \). Define \( \text{HilbPrf}_\alpha(x, y) \) to have a Concise Gödel Encoding iff \( \text{HilbPrf}_\alpha(x, y) \)'s formula is \( \Delta_0^- \) and there exists a constant \( R > 0 \) and an accompanying finite subset \( F \subset \alpha \) which satisfy the following two invariants:

I) \( \forall p \ \forall t \quad \text{If } p \text{ is the proof of the theorem } t \text{ from the axiom system } \alpha \text{ then } \exists q < 2^n \quad \text{where } q \text{ is a proof from axiom system } F \text{ of:} \)

\[
\forall x \ \forall y \quad \{ \sigma_t(x) \land \sigma_p(y) \implies \text{HilbPrf}_\alpha(x, y) \} \tag{31}
\]

II) Peano Arithmetic can formally prove that the triple \( (\alpha, F, R) \) satisfies Item I’s requirements. (This added condition is typically trivial to satisfy).

It is easily verified that essentially all r.e. axiom systems have concise encodings. Indeed, Definition 1’s constraint \( q < 2^n \) is somewhat excessive because \( q \) will typically have a much smaller magnitude. Theorem 4’s requirement that \( \alpha \) has a concise encoding is thus a very minor constraint.

It should also be noted that Definition 1’s formalism will be omitted when Appendix C sketches a proof of a somewhat stronger version of Theorem 4.

**Lemma 2.** Let \( \alpha \) denote any axiom system that can prove all Peano Arithmetic’s \( \Pi_1^- \) theorems and which also employs the multiplicative naming convention’s axioms. Let \( \sigma_n(x) \) again
be defined by Equation (16). Then there will exist a constant $K_\alpha$ (whose value depends only on $\alpha$) such that there exists a proof from $\alpha$ of the sentence (32), whose length is bounded by $K_\alpha + O(n^3)$.

$$\exists z \exists w \sigma_n(w) \land \text{LogLog}(z) \geq w$$

(32)

Proof. For $i \geq 2$ the multiplicative convention’s axiom $b_i$ (defined in Equation (14)) indicates $C_{i+1} = C_i \ast C_i$. Thus, the combination of the axioms $b_1, b_2, b_3 ... b_{n+1}$ imply that $C_{n+2} = 2^{2^n}$. Hence, there exists some finite subset of Peano Arithmetic’s $\Pi_1$ theorems, called say $F$, such that the union of $F$ with $b_1, b_2 ... b_{n+1}$ provides a proof of Equation (32), such that this proof has no more than $O(n^2)$ lines and each line uses $O(n)$ or fewer bits. This proof’s total bit-length will thus be bounded by $O(n^3)$.

The full proof of (32) will thus have two parts. Its first half will use $\alpha$ to prove all the theorems of $F$. Its length is represented by a constant $K_\alpha$ (whose value depends on $\alpha$). The second half, outlined in the preceding paragraph, will have an $O(n^3)$ size. Hence, the full proof has a $K_\alpha + O(n^3)$ length. □

Remark 3. The analog of Lemma 2 is false if the additive naming convention replaces the multiplicative convention in this lemma’s hypothesis. In particular, every proof of the existence of an integer $N$ using the additive convention will have a larger Gödel number than $N$. In contrast, the multiplicative convention allows for the Gödel number of such proofs to have a sharply smaller magnitude than $N$ (when for instance $N = 2^{2^k}$ and $k$ is larger than some fixed constant). In essence, this difference in magnitude is the reason the Second Incompleteness Theorem will generalize under the multiplicative naming paradigm — while Section 4 showed boundary-case exceptions to it exist under the additive naming convention.

Definition 2. Consider an encoding of the integer $N$ that uses only the constant symbols from the multiplicative naming convention and only the grounding functions of division and subtraction to encode $N$. There are of course an infinite number of such encodings for $N$. In this section, $\widehat{N}$ will denote the encoding of $N$ with the minimal such Gödel number (using Appendix A’s notation). For instance since the constant-symbols of $C_1$, $C_2$ and $C_5$ represent the integers of 1, 2 and 256 under the multiplicative naming convention, the value $\widehat{125}$ is encoded as: $[C_5 \div C_2] - C_1 - C_2$.
**Lemma 3.** For any integer $N$, the bit-length for the encoding of $\hat{N}$ has an $O(\log N^2)$ or smaller magnitude.

**Proof Sketch:** We will not provide a detailed justification of Lemma 3 here because it is both quite straightforward and actually not technically needed to prove Theorem 4. (The latter is because if one replaces Lemma 3’s exponent 2 with any larger fixed positive constant, then a sufficient bound will be available to prove Theorem 4). An abbreviated justification of the underlying methodology needed to prove Lemma 3 is given below:

1. If $N$ is a power of 2, then $\hat{N}$ can be encoded by taking the multiplicative naming convention’s next higher constant and dividing it by $\log\log(N)$ or fewer smaller named symbols.

2. Since any integer $N$ can be encoded by taking the power of 2 that is larger than it and then subtracting $\log(N)$ or fewer smaller powers of 2, it follows from Item 1 that each integer $N$ can be encoded using no more than $O(\log(N) \cdot \log\log(N))$ appearances of named constant symbols.

3. Because the $i$-th named constant symbol under Appendix A’s convention is encoded with $O(\log(i))$ bits and because Items 2’s constants $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ have indices satisfying $i_j \leq \log(N)$, it follows that no more than $O(\log^2(N))$ bits are needed to encode $\hat{N}$. □

**Definition 3.** Let $\log^\lambda z$ again denote $\log(\log(\log(...(\log(z))))))$ — where there are $\lambda$ iterations of the logarithm function. Section 2 had defined $\text{ShortPrf}_{\alpha, D}(x, y, z)$ to be a $\Delta_0$ formula indicating $y = \log^\lambda z$ and that $y$ is a proof of the theorem $x$ from the axiom system $\alpha$ using the deduction method $D$. Whenever the subscript $D$ is absent in this notation (as for example in the formula “$\text{ShortPrf}_{\alpha}^\lambda(x, y, z)$”), our default assumption shall be that $D$ denotes Hilbert deduction. Also likewise, the symbol $\bar{\Upsilon}^\lambda(\alpha)$ will be an abbreviation for Section 2’s Generalized Gödel Sentence $\bar{\Upsilon}^\lambda_D(\alpha)$ (defined in its sentence **) — where again the omitted symbol $D$ is taken by default to represent Hilbert Deduction. Translated into the English language, $\bar{\Upsilon}^\lambda(\alpha)$ will thus be an abbreviation for the following sentence:

In a context where one employs the “$\text{ShortPrf}_{\alpha}^\lambda(x, y, z)$” proof-notation, there exists no code $(y, z)$ that “proves” this sentence (looking at itself).
It is easy to assign $\mathcal{U}^\lambda(\alpha)$ a formal $\Pi_1^-$ encoding by following Gödel’s classic example. Thus, let $\text{Subst}^*(g, h)$ denote the following $\Delta_0^-$ formula:

$$\text{Subst}^*(g, h) = \{ \text{The integer } g \text{ is an encoding of a formula, and } h \text{ encodes a sentence identical to } g, \text{ except that all free variables in } g \text{ are now replaced with Definition 2’s term } \widehat{g} \}. $$

Then $\mathcal{U}^\lambda(\alpha)$ is defined as the $\Pi_1$ sentence $\Gamma(\widehat{n})$, where $\Gamma(g)$ denotes the formula in (33), $n$ is its Gödel number, and $\widehat{n}$ is a formal term representing $n$’s integer value.

$$\forall h \forall y \forall z \{ \text{Subst}^*(g, h) \Rightarrow \neg \text{ShortPrf}^\lambda_\alpha(h, y, z) \}$$

(33)

In Lemma 4, the symbol $\bot$ denotes the Gödel number of the sentence 0=1.

**Lemma 4.** Let $\alpha$ denote any axiom system satisfying Definition 1’s concise-encoding property. Suppose $\alpha$ can prove all Peano Arithmetic’s $\Pi_1^-$ theorems, and $\alpha$ additionally includes all the multiplicative naming convention’s axioms. Then there will exist some corresponding constant $L_\alpha$ such that Definition 3’s two formulae $\text{ShortPrf}^2_\alpha(x, y, z)$ and $\mathcal{U}^2(\alpha)$ will automatically satisfy the following invariant under the standard model of the natural numbers.

$$\forall z > \widehat{L_\alpha} \forall y \{ \text{ShortPrf}^2_\alpha(\widehat{\mathcal{U}^2(\alpha)}, y, z) \Rightarrow \exists x < z \text{ HilbPrf}_\alpha(\bot, x) \}$$

(34)

**Proof:** Consider a sentence identical to (34), except that its clauses $z > \widehat{L_\alpha}$ and $x < z$ are now removed. (Equation (35) illustrates this sentence.) We will first prove (35)’s validity in the standard model and then prove Lemma 4.

$$\forall z \forall y \{ \text{ShortPrf}^2_\alpha(\widehat{\mathcal{U}^2(\alpha)}, y, z) \Rightarrow \exists x \text{ HilbPrf}_\alpha(\bot, x) \}$$

(35)

We will prove (35) by assuming $(y, z)$ satisfies $\text{ShortPrf}^2_\alpha(\widehat{\mathcal{U}^2(\alpha)}, y, z)$. Our proof will construct four substrings, called $x_1, x_2, x_3, x_4$, such that their bit-wise concatenation represents the proof $x$ whose existence is claimed by (35). The formal definition of these four stings $x_1, x_2, x_3, x_4$ is as follows:

1. The substring $x_1$ will be simply the integer $y$. It will thus be a proof from $\alpha$ of $\mathcal{U}^2(\alpha)$.
2. The substring $x_2$ will be a *proof from $\alpha$ that $y$ proves $\mathcal{U}^2(\alpha)$*. (Since Lemma 4’s hypothesis contains a conciseness assumption, Definition 1 thereby implies $x_2 < 2^{yR}$, for a constant $R$ whose value depends on $\alpha$.)

26
3. The substring $x_3$ will be a proof from $\alpha$ that $\exists v \log \log(v) = \widehat{y}$. (Lemma 2 assures $x_3$ is sufficiently small for $\log(x_3) \leq K_\alpha + O(y^3)$, where $K_\alpha$ is a constant that depends only on $\alpha$.)

4. The substring $x_4$ will combine the intermediate results from the preceding segments to derive the conclusion "0=1". It is easy for $x_4$ to do this because $x_1$ proved the diagonalizing theorem $U^2(\alpha)$, which states essentially that "There is no code $(y,z)$ that is a proof of me", while $x_2$ and $x_3$ proved precisely such $y$ and $z$ do exist. (The length of $x_4$ is inconsequential because it is much smaller than the lengths of $x_1$, $x_2$ and $x_3$.)

This construction clearly shows that Equation (35) is valid under the standard model of the natural numbers because the integer $x$, that is the natural concatenation of the four substrings $x_1, x_2, x_3, x_4$, obviously satisfies the claim of Equation (35). Moreover, it is easy to extend our construction to also verify Equation (34)'s slightly stronger claim. This is because we can choose a constant $L_\alpha$ such that if $z > L_\alpha$ then each of the four preceding strings satisfy $\log(x_i) < \frac{1}{5} \log(z)$ (see footnote 2). Thus the natural concatenation of these four strings will consist of an element $x$ that meets Equation (34)'s requirement for satisfying $x < z$. □

**Corollary 1.** The sentence (34) is also a theorem of Peano Arithmetic.

**Proof Sketch:** The definition of Conciseness had included the Invariant II precisely so we could prove the current corollary. In particular, the Invariant II states that the Invariant I is provable from Peano Arithmetic (in addition to being logically valid). As a result, Lemma 4’s proof can be carried out in Peano Arithmetic. (The exact place we need Invariant II in the preceding construction is in its step 2.) The other steps in Corollary 1’s proofs are the same as their counterparts in Lemma 4’s proof; they need no justification here. □

**Lemma 5.** Suppose $\alpha$ is a consistent axiom system that satisfies Lemma 4’s hypothesis. Then Peano Arithmetic is able to prove the following $\Pi^1_1$ sentence.

$$\forall z \forall y \{ \text{ShortPrf}_\alpha^2 (\widehat{U^2(\alpha)}, y, z) \Rightarrow \exists x < z \text{ HilbPrf}_\alpha (\perp, x) \}$$

Items 1–4 of the footnoted paragraph specified upper bounds on the lengths of the four strings $x_1, x_2, x_3$ and $x_4$. It also may be presumed $y = \log \log(z)$ because $(y,z)$ satisfies the predicate $\text{ShortPrf}_\alpha^2 (\widehat{U^2(\alpha)}, y, z)$. These observations trivially imply that we can choose a large enough value for $L_\alpha$ such that each of the four $x_i$ automatically satisfy $\log(x_i) < \frac{1}{5} \log(z)$ when $z > L_\alpha$. 27
Proof. An intermediate step in our proof of Lemma 4 had shown Equation (35) was valid under the standard model. Since $\alpha$ is consistent, it immediately follows that (37) is also certainly valid under the standard model.

$$\forall y \forall z \neg \text{ShortPrf}^2_\alpha \left( \left[ \hat{\Omega}^2(\alpha) \right] , y, z \right)$$

(37)

The validity of (37) is of course insufficient to assure that Peano Arithmetic can actually derive this sentence as a theorem. However, it is well-known that Peano Arithmetic can prove every $\Delta_0$ sentence that is valid. In particular, Equation (38) (below) is a $\Delta_0$ sentence that differs from (37) by having the range of its universally quantified variables bounded by Lemma 4’s constant $L_\alpha$. Hence, the validity of (37) implies the validity of (38), which in turn implies that Peano Arithmetic must be able to prove (38) as a theorem.

$$\forall y \leq \hat{L}_\alpha \forall z \leq \hat{L}_\alpha \neg \text{ShortPrf}^2_\alpha \left( \left[ \hat{\Omega}^2(\alpha) \right] , y, z \right)$$

(38)

Since ShortPrf$_\alpha^2(x, y, z)$ forces $y < z$, Peano Arithmetic can use (38) to get:

$$\forall z \leq \hat{L}_\alpha \forall y \neg \text{ShortPrf}^2_\alpha \left( \left[ \hat{\Omega}^2(\alpha) \right] , y, z \right)$$

(39)

The combination of Corollary 1 and the preceding paragraph demonstrate that Peano Arithmetic can prove the assertions of Equations (34) and (39). This implies Peano Arithmetic can also formally verify Equation (36) (because it is an immediate consequence of (34) and (39)).

\[ \Box \]

**Theorem 5** Assume ShortPrf$_\alpha^\lambda(x, y, z)$ and Definition 3’s formula Subst*(g, h) have $\Delta_0$ encodings. Suppose $\alpha$ is an axiom system that has a capacity for proving all the $\Delta_0$ sentences that are valid in the standard model of the natural numbers. Also assume (for some fixed constant $\lambda$) that $\alpha$ has a capacity to prove the three theorems listed below. Then $\alpha$ is inconsistent.

A) $\ \\forall p \rightarrow \text{HilbPrf}_\alpha \left( \bot , p \right)$

B) $\ \\{ \exists y \exists z \ \text{ShortPrf}^\lambda_\alpha \left( \left[ \hat{\Omega}^\lambda(\alpha) \right] , y, z \right) \} \Rightarrow \exists x \ \text{HilbPrf}_\alpha \left( \bot , x \right)$

C) $\ \\forall g \ \forall h \ \forall h^* \ \{ \left[ \text{Subst}*(g, h) \land \text{Subst}*(g, h^*) \right] \Rightarrow h = h^* \}$

Justification. We have put the proof of Theorem 5 into Appendix B because its justification is quite similar to the Theorem 2.3 that was proved in [47].
Let us recall that Theorem 4’s formal statement (from Section 1) stated that no consistent axiom system $\alpha$ can prove a theorem corroborating its own Hilbert-Consistency when it contains all the multiplicative naming convention’s axioms, satisfies Definition 1’s “concise encoding” property and also retains an ability to prove all Peano Arithmetic’s $\Pi^1_1$ theorems. The formal proof of this theorem is given below:

**Proof:** For the sake of constructing a proof-by-contradiction, let us temporarily assume Theorem 4 was false. Then there would exist a consistent axiom system $\alpha$ satisfying Theorem 4’s hypothesis and the condition below:

$$\alpha \vdash \forall x \neg \text{HilbPrf}_\alpha (\bot, x) \quad (40)$$

Let $\lambda = 2$. It is then easy to establish $(\alpha, \lambda)$ satisfies the three requirements of Theorem 5’s hypothesis. The justification of this claim is given below:

1. Equation (40) shows $\alpha$ can prove Theorem 5’s needed sentence (A).

2. It is easy to apply Lemma 5 to establish $\alpha$ can also prove the sentence (B) from Theorem 5’s hypothesis. This is because Lemma 5 shows that Equation (36) is a theorem of Peano Arithmetic, and Theorem 4’s hypothesis indicated that $\alpha$ had a capacity to prove all Peano Arithmetic’s $\Pi^1_1$ theorems. Hence, $\alpha$ can prove the validity of Equation (36), which in turn implies $\alpha$ can prove Theorem 5’s sentence (B) (because the latter with $\lambda = 2$ follows from Equation (36)).

3. Since Theorem 5’s sentence (C) is another of Peano Arithmetic’s $\Pi^1_1$ theorems, $\alpha$ can clearly prove (C).

Thus $\alpha$ satisfies Theorem 5’s three requirements, and this theorem then implies that $\alpha$ must be inconsistent. This latter observation completes our proof-by-contradiction because it began by assuming $\alpha$ was consistent. □

**A Slightly Stronger Form of Theorem 4:** For the sake of keeping our presentation in this section reasonably short, Theorem 4 required its axiom systems $\alpha$ retain an ability to prove all Peano Arithmetic’s $\Pi^1_1$ theorems. The Appendix C will sketch a proof of a more elaborate form of Theorem 4, called Theorem 4*, where $\alpha$ is neither required to prove all Peano Arithmetic’s $\Pi^1_1$ theorems nor to be concisely-encoded. Instead, its incompleteness result will state that *one can isolate one single* $\Pi^1_1$ theorem $W$ of Peano Arithmetic, such that
Alternative Naming Conventions: The three most natural naming conventions are clearly the incremental, additive and multiplicative conventions, defined by Equations (12)–(14). It is also possible to consider hybridized naming conventions that lie midway between the additive and multiplicative conventions. For a fixed constant $H$ and any $i \geq 3$, let $\text{Hybrid}(H)$ refer to a naming convention that defines $C_i$ to equal $\lceil 2^\lfloor \log(i) \rfloor H \rceil \cdot C_{i-1}$. This convention can unify the formalisms of Theorems 3 and 4. Thus, Theorem 3’s partial exception to the Second Incompleteness Theorem remains valid when one sets $H = 1$ under the Hybrid convention. Similarly by choosing a constant $H > 1$, the second incompleteness effects under Theorems 4 and 4* can be generalized. This paper has focused on the additive and multiplicative naming conventions because they shorten the proofs for Theorems 3 and 4, and their naming methodologies are also very natural.

Added Comment. A more elaborate version of the proofs appearing in this section and Appendix C can generalize these incompleteness results from Hilbert-style deduction to cut-free deductive methods, such as semantic tableaux and Herbrand deduction.

6 Infinite Far Reach

An axiom system $\alpha$ is called Infinitely Far-Reaching iff there exists a finite subset of axioms $S \subset \alpha$ such that for arbitrary $N$ this fixed-and-finite set $S$ is sufficient to prove $\exists x \text{Pred}_N(x) = 1$. This section will demonstrate Infinitely Far-Reaching systems exist that are capable of recognizing their own Hilbert consistency. The ISINF($A$) system, defined in this section, will possess an unnatural quality because it will be Infinitely Far-Reaching without sustaining an ability to prove successor is a total function. Nevertheless as a theoretical albeit highly artificial instrument, ISINF($A$) is useful because of its counter-intuitive nature.

Notation Used in this Section: As earlier in this paper, $\text{HilbPrf}_{\alpha}(x, y)$ will denote that $y$ is a Hilbert-style proof of $x$ from $\alpha$. In our current discussion, $\text{JumpPrf}_{\alpha}(x, y)$ will denote a second $\Delta_0^-$ predicate, specifying that either the bit-string encoding the integer $2y$ or the bit-string encoding $1+2y$ represents a Hilbert-style proof from $\alpha$ of the theorem $x$. A key point is that if a $\Delta_0^-$ formula can identify $\alpha$’s axioms, then $\text{JumpPrf}_{\alpha}(x, y)$ will have a $\Delta_0^-$ encoding, even if $\alpha$ does not technically recognize either successor or the operation...
of doubling are formally total functions. (In essence, ISINF(A)’s counter-intuitive — albeit artificial — features will be due to this fact.)

**Definition of ISINF(A):** The system ISINF(A) will contain four axiom groups, similar to ISCE(A). Its Group-zero axiom will simply indicate the existence of the first three natural numbers, called 0, 1 and 2. ISINF(A)’s Group-1 and Group-2 axioms will have the same definitions as they did under ISCE(A). The novel aspect of ISINF(A) will be its Group-3 scheme. Unlike our prior formalism, ISINF(A)’s Group-3 scheme will contain two axioms. These axioms, formalized by Equations (41) and (42), will be defined *simultaneously* via the self-referencing methodology of the Fixed Point Theorem. In essence, (41)’s fixed-point axiom will assert the ISINF(A) formalism is Hilbert-consistent. On the other hand, Theorem 6 will prove that (42)’s axiom will make ISINF(A) Infinitely Far Reaching. (The symbol \( \lceil \frac{x}{2} \rceil \) in Equation (42) will denote \( x \) divided by 2 with upwards rounding, and \( \bot \) will denote the Gödel number of the sentence 0=1.)

\[
\forall x \quad \neg \text{HilbPrf}_{\text{ISINF(A)}}(\bot, x) \quad (41)
\]

\[
\forall x \quad \{ \quad \forall y \leq \lceil \frac{x}{2} \rceil \quad \neg \text{JumpPrf}_{\text{ISINF(A)}}(\bot, y) \quad \Rightarrow \quad \exists z \quad x = z - 1 \quad \} \quad (42)
\]

We will not provide a formal encoding for (41) and (42)’s axioms because Lemma 1’s treatment of ISCE(A)’s Group-3 axiom has direct analogs for both of these two axioms. Thus, \( \text{HilbPrf}_{\text{ISINF(A)}}(\bot, x) \) and the formula \( \text{JumpPrf}_{\text{ISINF(A)}}(\bot, y) \) both have \( \Delta_0^- \) encodings.

**Theorem 6.** Let \( A \) denote an arbitrary axiom system that satisfies the invariant conditions of:

i. Each of \( A \)’s \( \Pi_1^- \) theorems is valid in the standard model, and

ii. The formula \( \text{HilbPrf}_A(x, y) \) employed by ISINF(A)’s Group-2 scheme will have a \( \Delta_0^- \) encoding.

Then ISINF(A) will be both 1) consistent and 2) Infinitely Far-Reaching.
Proof of Item 1. Suppose for the sake of establishing a proof-by-contradiction that item 1 was false. Then one could construct a system $A$ satisfying conditions (i) and (ii) where ISINF($A$) is inconsistent. In this context, let $p$ denote the minimal-sized proof of $\bot$ from ISINF($A$).

Our contradiction proof will have a structure similar to Theorem 3’s proof. We will therefore only sketch it. The only method by which ISINF($A$) can learn about the existence of any integer $i \geq 3$ consists of repeatedly applying Equation (42)’s “Expansion Axiom” to successively construct the integers $3, 4, 5, \ldots i$. Thus if $p$ is the smallest proof of $\bot$, then $j = 2 \cdot \lceil \frac{p}{2} \rceil - 1 \leq p - 1$ is the largest integer whose existence is implied by axiom (42).

Let $M_j$ denote the finite model which assumes that the only integers which exist are the numbers $0, 1, 2, \ldots j$. Since the previous paragraph indicated $j = 2 \cdot \lceil \frac{p}{2} \rceil - 1$ where $p$ is the minimal proof of $\bot$, the axiom (42) must be valid in the model $M_j$. Also, all ISINF($A$)’s Group-zero, 1 and 2 axioms must be valid in the model $M_j$, by the same reasoning as was used in Theorem 3’s proof (see footnote 3).

The main point is that since $p$ is a proof of $\bot$, some axiom in $p$’s proof must be invalid in the model $M_j$. This implies axiom (41) is invalid in $M_j$ (because each other axiom used in $p$’s proof is valid in $M_j$). This contradicts $p$’s presumed minimality by implying that a smaller $q \leq j < p$ is also a proof of $\bot$. Hence Item 1 is true because its negation is contradictory. □

Proof of Item 2. Essentially a trivial consequence of the combination of Item 1 with axiom (42)’s formal structure. In particular, let $\phi(x)$ denote the expression enclosed by the square brackets of (42). This expression is $\Delta^0_\pi$ because JumpPrf$_{\text{ISINF}(A)}$( $\bot$, $y$) was. The key point is that every logically valid $\Delta^0_\pi$ formula $\gamma(c)$ has the property that ISINF($A$) is capable of proving $\gamma(c)$ for any integer $c$ whose formal existence is known to ISINF($A$). Thus, (42)’s square bracket expression has this property. Hence ISINF($A$) can combine axiom (42) with $\phi(c)$’s verification property to automatically infer the existence of a larger constant $c + 1$ from the existence of $c$. Thus, ISINF($A$) is Infinitely Far-Reaching because it can learn of the existence of any natural number $n$ by applying $n - 2$ iterations of the preceding rule. □

Section 4 defined “2-reduced” $\Pi^1_1$ sentences and noted every “2-reduced” $\Pi^1_1$ sentence is automatically valid in each finite model $M_j$ having $j \geq 2$ when it is valid in the standard model of the natural numbers. Since each of ISINF($A$)’s Group-zero, 1 and 2 axioms are such 2-reduced $\Pi^1_1$ sentences, their validity under $M_j$ is thus immediately assured.
Generalizations of Theorem 6 and Its Significance. The system ISINF(A) is awkward because of the appearance of the two different proof predicates “HilbPrf_α (x, y)” and “JumpPrf_α (x, y)” in its Group-3 axioms. A second drawback is that ISINF(A) appears to be incompatible with the tangibility reflection principles (defined at the end of Section 4). It is thus quite unlike ISCE(A), which was fully compatible with the tangibility reflection principles (via Theorem 3*). As a whole, ISCE(A) is thus preferable over ISINF(A). One reason ISINF(A) is partially interesting, despite its awkward nature, is that it seems to clarify the meaning of the Theorems 1, 2 and 4, by Pudlák, Solovay and ourselves. In particular, it shows that these generalizations of the Second Incompleteness theorem do not technically apply to all formal axiom systems with Infinite Far Reach.

There is also a second interesting facet of Theorem 6’s formalism that was brought to our attention in the form of an open question from Pavel Pudlák. Let us call \( F(x) \) an Extender Function iff it satisfies axioms (43) and (44).

\[
\forall x \forall y \quad x \neq y \implies F(x) \neq F(y) \tag{43}
\]

\[
\forall x \quad F(x) \neq 0 \tag{44}
\]

It is easy to construct a finite set of \( \Pi_1^- \) axioms, called say \( S \), such that the union of \( S \) with the two axioms from Equations (43) and (44) constitutes an axiom system of Infinite Far Reach. The study of axiom systems that employ such extender functions was initiated by Ajtai [4] during his investigation of various generalizations of the Pigeon Hole Principle. In this context, Pudlák’s question was whether one could use an Extender Function to construct a self-justifying axiom system of Infinite Far Reach?

In order to present one formalized version of this open question, let us define IS.Extender(A) to be an axiom system identical to Section 4’s ISCE(A) formalism except for the following changes:

1. The IS.Extender(A) formalism will contain an extra function symbol \( F \), and its Group-1 axiom class will contain the two additional axiom-sentences listed in Equations (43) and (44). (No change will be made among IS.Extender(A)’s Group-2 axioms; they will thus not discuss any further properties of the newly created function symbol \( F \)).

2. The Group-3 axiom of IS.Extender(A) will be identical to ISCE(A)’s Group-3 Kleene-like “I am consistent axiom” except that the pronoun “I” will now refer to a revised
system that contains the two additional axiom-sentences given in Equations (43) and (44).

Based on the fact that Theorems 3, 3* and 6 discuss three close cousins of IS.Extender(A) that are automatically consistent when $A$ is consistent, it is reasonable to conjecture that IS.Extender(A) has a similar consistency property. Pavel Pudlák, in private communications, noticed that if IS.Extender(A) satisfied this consistency condition, then it would be a significant generalization of Theorem 6’s self-justifying system. (This is because IS.Extender(A) employs a more natural form of Infinite Far Reach than ISINF(A).) The purpose of this paragraph is thus to bring to the research community’s attention this interesting question, raised by Pudlák.

For the sake of clarity, it should be noted that if one were to supplement the Equations (43) and (44) with Equation (45)’s further axiom, then a generalization of the Second Incompleteness Theorem will be activated.

$$\forall x \forall y \ x < y \Rightarrow F(x) < F(y)$$ (45)

In particular, Theorem 7 formalizes this effect.

**Theorem 7.** There exists a $\Pi_1^-$ theorem $W$ of Peano Arithmetic such that no consistent axiom system $\alpha$ can simultaneously prove $W$, prove the validity of the sentences in Equations (43) through (45) and prove a theorem verifying its own Hilbert consistency.

The Appendix D sketches a proof for Theorem 7 that essentially corroborates Theorem 7 by a reduction argument to the earlier results of the Theorems 1 and 2 by Pudlák and Solovay [25, 32]. As Appendix D explains, there is one added complication in proving Theorem 7, which did not arise in the proofs of the Theorems 1 and 2 by Pudlák and Solovay. After overcoming this problem, Appendix D will then finish Theorem 7’s proof by applying the earlier methods that [25, 32] had previously used to prove their Theorems 1 and 2.

Theorem 7 is interesting partly because it shows that axiom systems only slightly stronger than IS.Extender(A) are known to obey the Hilbert-style version of the Second Incompleteness Theorem. Thus if IS.Extender(A) turns out to be a self-justifying axiom system, then it will represent close to the maximal type of such a formalism that is feasible under at least Hilbert style deduction. An interesting open question to determine is where between Theorem 6’s boundary-case exception to the Second Incompleteness Theorem and Theorem 7’s generalization of it does the Is.Extender(A) formalism lie?
7 Concluding Remarks

The sharp contrast between the positive and negative results of Theorems 3 and 4 is the main result of this paper. In combination, these propositions indicate the Second Incompleteness Effect for Hilbert-style deduction can be evaded when an axiom system employs the additive naming convention, although it becomes operative when the multiplicative convention is present.

These results differ from our prior study of semantic tableaux deduction because [43, 46, 48, 51] used axiom systems that could simultaneously recognize their own semantic tableaux consistency and addition as a total function. On the other hand, the Theorem 2 by Solovay [32] showed that no analog of this paradigm for Hilbert-styled deduction can even recognize successor as a total function. To evade the obstacle that was formalized by Solovay’s theorem, our ISCE(A) formalism has thus replaced the conventional axiom declaring addition is a total function with an additive naming convention.

Section 4’s ISCE(A) axiom system is of interest also because it satisfies a “continuous expansion property” — where the concerned axiom system is associated with a sequence of integers $K_1 < K_2 < K_3 \ldots$ such that the union of all the axioms with Gödel number less than $K_i$ can be combined to prove the existence of an integer greater than $K_{i+1}$. This construct is weaker than an axiom declaring successor is a total function. However, it allows ISCE(A) to retain at least some weak notion of an infinite growth among integers. Indeed, Equation (29) has shown that ISCE(A)’s continuous expansion sequence grows at a very fast super-exponential rate. Our chief results are thus that this rapid-growth of ISCE(A)’s additive convention is compatible with a boundary-case exception to the Second Incompleteness Theorem for Hilbert-style deduction, while the stronger multiplicative naming conventions (of Theorems 4 and 4*) will reactivate the Second Incompleteness Theorem.

Acknowledgments: I thank Robert Solovay for several telephone conversations we had during 1994. During those conversations (see [32]), Solovay described the Theorem 2, whose statement appeared in the Introduction Section and which stimulated much of the research in this paper. I thank Pavel Pudlák for his comments during 2001 that were summarized in Section 6, and which had stimulated my discovery of Theorem 7. I also thank the anonymous referee for several very useful comments about this manuscript.
Appendix A: Summary of Gödel Encoding Method

This appendix will briefly summarize our formal method for generating Gödel numbers for logical sentence and proofs. This encoding scheme will be roughly analogous to the natural B-adic encoding methods described by Hájek-Pudlák [11] and Wilkie-Paris [42] — insofar as the number of utilized bits to encode a semantic object will be approximately proportional to the length of such an expression written by hand.

Our encoding scheme will use the following 20 language symbols to formalize a logical sentence or proof:

1. The standard connective symbols of $\land$, $\lor$, $\neg$, $\Rightarrow$, $\forall$ and $\exists$.
2. The left and right parenthesis symbols, and also a comma symbol.
3. Seven function symbols for representing the seven grounding functions of subtraction, division, logarithm, etc.
4. The relation symbols of “$=$” and “$\leq$”.
5. The symbol $\hat{V}$ for designating the presence of a basic variable symbol.
6. The symbol $\hat{C}$ for designating the presence of a constant symbol $C$.

Let a byte denote an unit of six bits. Define a proof as being either a sequence of bytes (or equivalently being an integer, written in base 64). Each of the 20 symbols (above) will be given some unique 6-bit code, ranging between 32 and 51. Our method for representing the presence of the i-th variable $v_i$ will be to encode it is as a string of $\lceil \log_{32}(i+1) \rceil + 1$ bytes, where the first byte is the “$\hat{V}$” symbol and the remaining bytes encode $i$ as a base-32 number. The same convention will be used to denote the presence of the i-th constant symbol $C_i$ except its first byte will be the “$\hat{C}$” symbol. (This method for encoding the name of the $i-th$ constant symbol in a digital-like format is the reason that Sections 1 and 3 had stated that the $i-th$ constant symbol’s name would have the important needed $O(\Log(i+1))$ bit-length.)

Our byte-styled encoding method will produce proof strings whose length is approximately proportional to the effort to write down such a proof by hand. There are many analogs of such types of encodings in the prior literature (see for example the Hájek-Pudlák textbook [11]).
Such compressed encodings are usually considered to be preferable and more efficient than an uncompressed encoding method, using say the Chinese Remainder Theorem [20]. All our theorems have analogs under such uncompressed encoding methods, but they are substantially more meaningful when one uses efficiently compressed encodings.

Appendix B: Sketch of Theorem 5’s Proof

Theorem 5’s formal statement is similar to article [47]’s Theorem 2.3, except that it discusses Hilbert deduction rather than semantic tableaux deduction. We will therefore provide only a sketch of Theorem 5’s proof.

Proof Sketch. Let $\mathcal{U}^*$ denote the “diagonalizing” sentence defined below:

$$
\mathcal{U}^* =_{df} \{ \forall y \forall z \neg \text{ShortPrf}^\lambda(\mathcal{U}^\lambda(\alpha), y, z) \}
$$

(46)

Consider the sentences (A) and (B) from Theorem 5’s hypothesis. From the definitions of (A) and (B), it is trivial that “$A \land B \vdash \mathcal{U}^*$”. Since (A) and (B) are provable from $\alpha$, we get:

$$
\alpha \vdash \mathcal{U}^* 
$$

(47)

It is clear that $\mathcal{U}^\lambda(\alpha)$ and $\mathcal{U}^*$ are equivalent sentences under sufficiently strong models of arithmetic. However, we need more than this fact to prove Theorem 5. We need establish that a weak axiom system $\alpha$ (satisfying Theorem 5’s hypothesis) can also recognize this equivalence.

To establish this last fact, let $\mathcal{U}^\lambda(\alpha)$ denote (33)’s Gödel number, encoded with Definition 2’s overbrace notation. Then $\mathcal{U}^\lambda(\alpha)$’s definition implies $\text{Subst}^*(\mathcal{U}^\lambda(\alpha))$ is true. This implies (48) because Theorem 5’s hypothesis indicates $\alpha$ can prove all logically valid $\Delta_0^-$ sentences.

$$
\alpha \vdash \text{Subst}^*(\mathcal{U}^\lambda(\alpha))
$$

(48)

We shall now use (48) to deduce the validity of the Equation (49) (given below). Let $\Lambda$ represent the sentence $\text{Subst}^*(\mathcal{U}^\lambda(\alpha))$, $\Theta$ represent the sentence (C) in Theorem 5’s hypothesis, and $\Xi$ be the identity $\mathcal{U}^\lambda(\alpha) \equiv \mathcal{U}^*$. For these $\Lambda$, $\Theta$ and $\Xi$, it is easy to infer that $\alpha \vdash \Lambda \land \Theta \Rightarrow \Xi$ (because $\Lambda \land \Theta$ enables $\alpha$ to immediately deduce that the only value for $h$ satisfying $\text{Subst}^*(\mathcal{U}^\lambda(\alpha), h)$ is the quantity $\mathcal{U}^\lambda(\alpha)$. Thus since $\alpha$ can
prove \( \Lambda, \Theta \) and \( \Lambda \land \Theta \Rightarrow \Xi \), we get:

\[
\alpha \vdash \overline{\Omega}^\lambda(\alpha) \equiv \overline{\Omega}^* \tag{49}
\]

Also, the combination of Equations (47) and (49) trivially implies that

\[
\alpha \vdash \overline{\Omega}^\lambda(\alpha) \tag{50}
\]

The justification of Theorem 5 will now be finished by applying the roughly classic paradigm where a proof formally verifies the statement that: “There is no proof of me”. In particular, let \( p \) denote the proof of \( \overline{\Omega}^\lambda(\alpha) \). (Note that \( p \)'s existence is assured by Equation (50).)

Choose a second integer \( q \) satisfying \( \text{Log}^\lambda(q) = p \). Let \( r \) denote the Gödel number of the sentence \( \overline{\Omega}^\lambda(\alpha) \). Also, if \( \overline{n} \) represents (33)'s Gödel number, then \( \overline{\Omega}^\lambda(\alpha) \) is encoded as:

\[
\forall h \forall y \forall z \{ \text{Subst}^\ast( \overline{n} \, , \, h ) \Rightarrow \neg \text{ShortPrf}^\lambda_\alpha( h \, , \, y \, , \, z ) \} \tag{51}
\]

The key point is that Equation (51) must be false because if one replaces its three variables \( y, z, \) and \( h \) with the three constants \( p, q, \) and \( r \) then (51)'s formal statement is clearly negated (via the usual diagonalization argument). Since Theorem 5’s hypothesis indicates \( \alpha \) has a capacity to prove all valid \( \Delta_0^- \) sentences, it clearly must also have a capacity to refute any \( \Pi_1^- \) sentence that is invalid in the Standard Model. Hence, we obtain:

\[
\alpha \vdash \neg \overline{\Omega}^\lambda(\alpha) \tag{52}
\]

The combination of (50) and (52) shows that \( \alpha \) is inconsistent. \( \square \)

Appendix C: An Alternate Version of Theorem 4’s Result

The Theorem 4* (proven in this appendix) will differ from Section 5’s Theorem 4 essentially by isolating a particular \( \Pi_1^- \) theorem \( W \) of Peano Arithmetic that serves as a threshold for activating the Second Incompleteness Theorem when the multiplicative naming convention is present. Our proof of Theorem 4* will be an incremental modification of Theorem 4’s proof. It will therefore be kept very brief. It will use the following notation:

1. An axiom system \( \alpha \) will be called \textbf{Finitely Generated} iff there exists a finite list of axioms \( \phi_1, \phi_2, \ldots, \phi_n \), such that \( \alpha \) consists of the union of these \( n \) axioms with the infinite set of axioms of \( b_1, b_2, b_3 \ldots \) employed by the multiplicative naming
convention. In this case, \textbf{Base}(\alpha) will denote the byte-sequence encoding the axiom list: “\(\phi_1, \phi_2, \phi_3, \ldots, \phi_n\)”. Also, \(A\) will denote Base(\alpha)’s “Gödel number”.

2. \textbf{FinGenPrf}(A, t, p) will denote a \(\Delta_0^-\) formula which states that \(p\) is a Hilbert-style proof of the theorem \(t\) from a finitely generated axiom system whose particular Gödel number is identified by the integer \(A\).

**Lemma 6.** There exists a \(\Pi_1^-\) theorem \(V\) of Peano Arithmetic and an exact formalization \(M\) for encoding FinGenPrf\((A, t, p)\) as a \(\Delta_0^-\) formula such that no consistent, finitely-generated axiom system \(\alpha\) may contain both \(V\) as an axiom and prove (53)’s statement corroborating its own Hilbert consistency.

\[
\forall p \; \neg \text{FinGenPrf}(\lfloor \text{Base}(\alpha) \rfloor, \bot, p)
\]  

(53)

Lemma 6’s proof is not provided here because the same basic techniques that Section 5 used to prove its Theorem 4 will also corroborate Lemma 6. Thus, the intuition behind Lemma 6 is quite simple: It is that if one is allowed the freedom to pick the most amenable available \(\Delta_0^-\) encoding for the formula FinGenPrf\((A, t, p)\), henceforth denoted as \(M\), and also given the freedom to choose a sufficiently strong accompanying \(\Pi_1^-\) sentence \(V\), then these two extra degrees of freedom will allow one to easily modify Section 5’s proof of Theorem 4 to additionally corroborate Lemma 6’s statement.

Lemma 6 will be interesting because it turns out that after one has established its validity, \textit{for just one specialized form of \(\Delta_0^-\) encoding of the FinGenPrf\((A, t, p)\) predicate}, a significantly more general result can be derived — whose formalism is independent of the utilized \(\Delta_0^-\) encoding and which also applies to essentially all axiom systems of infinite cardinality as well. One further definition will help explore this point.

**Definition 4.** Let \(\beta\) denote a finitely-generated axiom system and \(\alpha\) denote a recursively enumerable axiom system that is not necessarily finitely-generated. In this context, \(\beta\) will be called a \textbf{Base-Subset} of \(\alpha\) iff all \(\beta\)’s axioms are axioms of \(\alpha\).

**Lemma 7.** There exists a \(\Pi_1^-\) sentence \(V^*\) such that the union of \(V^*\) with the multiplicative naming convention’s axioms can prove Equation (54) whenever (using Definition 4’s notation) \(\beta\) is a base-subset of \(\alpha\).
∀t ∀p \ [ \text{FinGenPrf}\left(\left[\text{Base}(\beta)\right], t, p\right) \Rightarrow \text{HilbPrf}_\alpha(t, p) \] \tag{54}

Definition 4 makes it quite easy to construct a $\Pi^1_1$ sentence $V^*$ which satisfies Lemma 7’s requirements. We will omit the construction of $V^*$ here because our goal is to keep this appendix very brief.

**Definition 5.** The symbol $W$ will denote a $\Pi^1_1$ sentence that is the conjunction of the two sentences $V$ and $V^*$ (defined by Lemmas 6 and 7).

**Theorem 4*.** (A strengthened version of Theorem 4’s result). No consistent recursively enumerable axiom system $\alpha$ which contains Definition 5’s axiom $W$ as well the multiplicative naming convention’s axioms can prove a $\Pi^1_1$ theorem indicating the non-existence of a proof of $0=1$ from itself.

**Proof:** For the sake of establishing a proof-by-contradiction, let us assume Theorem 4* was false. Then some consistent recursively enumerable axiom system $\alpha$ will satisfy Theorem 4* ’s hypothesis and be able to prove its own Hilbert consistency.

Let $p$ denote such a proof of $\alpha$’s consistency, and $\gamma$ denote the finite subset of axioms from $\alpha$ that appears in the proof $p$. Also, let $\beta$ denote the union of $\gamma$ with the combination of the $\Pi^1_1$ sentence $W$ and with all the multiplicative naming convention’s axioms.

Then $\beta$ is a finitely-generated system (because $\gamma$ had finite cardinality). Since the first paragraph of this proof had presumed $\alpha$ was consistent and because $\beta \subset \alpha$, it follows that $\beta$ must also be certainly consistent. Hence, $\beta$ satisfies the two requirements of Lemma 6’s hypothesis. This lemma will thus imply that $\beta$ is unable to prove its own Hilbert consistency.

We will now complete our proof-by-contradiction by proving the statement $+$ below (which contradicts the preceding paragraph’s final sentence):

$+$ The axiom system $\beta$ can verify its own Hilbert-consistency.

We will use Lemma 7 to prove $+$. This lemma implies that any finitely-generated axiom system, containing $V^*$, has the ability to infer $\beta$’s consistency from $\alpha$’s consistency. Also, $V^*$ is provable from $\gamma$ because $\gamma$’s definition indicated it includes the axiom $W$ (and because Definition 5 indicated that $W$ includes the clause $V^*$.) These facts imply that $\beta$ must be able to verify its own Hilbert-consistency because $\gamma$ (which is a subset of $\beta$) possesses an ability to verify the consistency of $\alpha$ (which is a superset of $\beta$). Hence, Lemma 7 certainly implies the validity of $+$. 

40
The combination of the preceding proofs of $+$ and of $+$’s negation enables our proof-by-contradiction to reach its desired end by showing an unavoidable contradiction will occur if Theorem 4* is false. □

**Added Comment:** The only significant difference between Theorems 4 and 4* is that the latter isolates a particular $\Pi^1_1$ sentence $W$ that acts as an *uniform threshold* for activating the multiplicative naming convention’s version of the Second Incompleteness Theorem. This appendix’s proof for Theorem 4* was written in a highly abbreviated style essentially because Section 5 had already proved the only slightly weaker Theorem 4.

Also, it should be noted that Theorem 4* ’s requirement that $\alpha$ contain the $\Pi^1_1$ sentence $W$ as an axiom can be replaced by a milder constraint that $\alpha$ merely retain an ability to prove some fixed and pre-specified $\Pi^1_1$ theorem $W^*$.

**Appendix D: Sketch of Theorem 7’s Proof**

This appendix will sketch a proof of Theorem 7 by employing the Theorems 1 and 2 of Pudlák and Solovay as the main interim step that is needed to corroborate Theorem 7. The key challenge in proving Theorem 7 can be realized when one considers the possibility that we could plausibly be examining a non-standard model $M$ of the set of natural numbers where the extender function $F$ satisfies the following two conditions:

1. $F(x) = x + 1$ when $x$ is a standard integer, and
2. $F(x) = x - 1$ when $x$ is any non-standard number.

Such a function $F$ is consistent with the axioms from Equations (43) through (45). However, this particular function $F$ (and many other examples) will grow at a slower rate than the successor function. Our strategy will thus be to find a way to apply the machinery of the Theorems 1 and 2 of Pudlák and Solovay to prove Theorem 7 despite the added complication that $F$ could be plausibly growing at a slower rate than the successor function.

To overcome this difficulty, let $\Psi(x)$ denote the following formula

$$\forall \ t \leq x \ F(t) > t$$

(55)
Also, let us assume that Theorem 7’s $\Pi_1^-$ sentence $W$ is a conjunction of several $\Pi_1^-$ clauses $W_1$, $W_2$, ... $W_n$ where its first three clauses $W_1$, $W_2$ and $W_3$ are defined below:

\[
\forall g \forall h \exists i \leq h \ [ g < h \Rightarrow g = i - 1 ] \tag{56}
\]

\[
\forall a \forall b \forall c \forall d \ [ a < b < d \wedge c - 1 = a ] \Rightarrow c < d \tag{57}
\]

\[
\forall a \forall b \forall c \ [ a \leq b \wedge b \leq c ] \Rightarrow a \leq c \tag{58}
\]

Then by essentially utilizing Equations (44), (45), (56), (57) and (58) as helpful intermediate steps, Theorem 7’s axiom system $\alpha$ can prove that the formula $\Psi(x)$ has the central properties of a definable cut. By this, we mean that $\alpha$ can prove that $\Psi(x)$ satisfies the following three conditions:

A. $\Psi(0)$

B. $\forall p \ \Psi(p) \Rightarrow \exists q \ [ \text{Predecessor}(q) = p \wedge \Psi(q) ]$

C. $\forall p \forall q \ [ q < p \wedge \Psi(p) ] \Rightarrow \Psi(q)$

The central point is thus that although the axiom system $\alpha$ may lack the power to recognize that successor is a total function in a formally global sense, it will be able to apply the conjunction of Equations (44), (45), (56) and (57) to infer the validity of the Item B (above) — which essentially states that the operation of successor is indeed a total function in a local sense among the set of integers $x$ that satisfy the condition $\Psi(x)$.

Once we have established the preceding condition, the remainder of Theorem 7’s proof can rest on a reduction argument that applies essentially the formal machinery that Pudlák and Solovay used to prove Theorems 1 and 2. By this we mean that if $\alpha$ could prove the sentence $\forall p \ \neg \text{HilbPrf}_\alpha(\lceil 0 = 1 \rceil, p)$ then it could clearly also verify Equation (59)’s assertion (which essentially states that no integer $p$ satisfying $\Psi(p)$ is a proof of $0=1$.)

\[
\forall p \ [ \Psi(p) \Rightarrow \neg \text{HilbPrf}_\alpha(\lceil 0 = 1 \rceil, p) ] \tag{59}
\]

Since items A–C establish $\Psi$ is functionally equivalent to a definable cut, the prior mathematical machinery from the literature about definable cuts can then finish the proof of Theorem 7, in the same manner that Pudlák and Solovay previously used it to prove Theorems 1 and 2. In particular, this method will establish that no consistent axiom system $\alpha \supset W$ can
prove its own consistency — because otherwise \( \alpha \) would then also prove Equation (59) — a result which would render \( \alpha \) automatically inconsistent.

In summary, this appendix was kept abbreviated because it sought to only summarize how the methods of Pudlák and Solovay can be used to prove Theorem 7’s statement, even in the extreme case where \( \alpha \) does not recognize successor as a total function. It has shown \( \alpha \) can prove \( \Psi(x) \) satisfies conditions A–C, thus making the Pudlák-Solovay paradigm applicable.

References


[32] R. Solovay, Private Communications (1994) about his generalization of one of Pudlák’s theorems from [25], using also the techniques of Nelson and Wilkie-Pairs [21, 42]. Solovay never published any of his observations about “Definable Cuts” that numerous logicians [7, 11, 14, 17, 21, 23, 25, 27, 42] have attributed to his private communications. The Theorem 2 (in Section 1) gave a 2-sentence summary of the result Solovay communicated over the telephone to us in April of 1994. A more detailed 4-page summary of our conversations with Solovay, including a type of proof of his theorem, appears in Appendix A of [46].


44


