Passive Induction and a Solution to a Paris-Wilkie Open Question

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Abstract

In 1981, Paris and Wilkie raised the open question about whether and to what extent the axiom system $I\Sigma_0$ did satisfy the Second Incompleteness Theorem under Semantic Tableaux deduction. Our prior work showed that the semantic tableaux version of the Second Incompleteness Theorem did generalize for the most common definition of $I\Sigma_0$ appearing in the standard textbooks.

However, there was an alternate interesting definition of this axiom system in the Wilkie-Paris article in the *Annals of Pure and Applied Logic* 35 (1987) pp. 261-302 which we did not examine in our year-2002 article in the *Journal of Symbolic Logic.* Our first goal is to show that the incompleteness results of our prior paper can generalize in this alternate context. We will also develop a formal analysis, using a new technique called *Passive Induction*, that is simpler than the formalism we had used before.

A further reason our results are of interest is that we have shown in a companion paper published in *Electronic Notes in Theoretical Computer Science* 165 (2006) pp. 213–226 that some *very unorthodox* axiomizations for $I\Sigma_0$ are anti-thresholds for the Herbrandized version of the Second Incompleteness Theorem. Thus, different axiomizations for $I\Sigma_0$ have nearly fully opposite incompleteness properties.

This paper is self-contained. It will not require a knowledge of our earlier results.

1 Introduction

This paper addresses an open question raised by Paris and Wilkie in 1981 [23]. It is well known that Gödel's Second Incompleteness Theorem [12] asserts that neither Peano Arithmetic nor any consistent extension of it can prove a theorem affirming its own consistency under Hilbert deduction. There have been numerous generalizations and extensions of Gödel's seminal result [1, 2, 3, 4, 5, 7, 9, 10, 14, 19, 16, 17, 22, 23, 24, 25, 26, 27, 28, 30, 33, 35, 36, 37, 38, 40, 42, 44]. For example, the combined work of Pudlák and Solovay [25, 30] has shown that essentially no

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axiom system that recognizes Successor$(x) \; = \; x+1$ as a total function can prove a theorem affirming its own consistency under Hilbert deduction.

In 1981, Paris and Wilkie [23] asked whether the axiom system IΣ_0, often also called IΔ_0, did satisfy the Second Incompleteness Theorem for cut-free methods of deduction, such as Semantic Tableaux. Interestingly, Wilkie-Paris [37] discovered IΣ_0+Exp is unable to prove the Hilbert consistency of the axiom system Q. Thus, the Paris-Wilkie open question pertains essentially to Semantic Tableaux and Herbrand deduction.

We will use a convention where a formula is called Δ_0 iff all its quantifiers are bounded quantifiers in the language of arithmetic (i.e. the quantifiers " $\forall \, v \, \leq \, T$ " and " $\exists \, v \, \leq \, T$ " use terms T that are built with only the addition and multiplication function symbols). A formula will be called Π_1 iff it can be written as $\forall \, v_1 \; \forall \, v_2 \; ... \; \forall \, v_n \quad \phi(v_1, v_2, ... v_n)$ where $\phi(v_1, v_2, ... v_n)$ is a Δ_0 formula.

Let us next recall that Tarski-Mostowski-Robinson [33] defined Q to be a *very weak* axiom system that defined addition and multiplication to be total functions *but contained no induction principle* or much other useful information about addition and multiplication. There are two natural ways for defining the IΣ_0 axiom system. It can be defined as using either Equations (1) or (2)'s limited induction scheme for Δ_0 formulae, combined with the Tarski-Mostowski-Robinson [33] axiom system, called Q.

$$\forall x \quad \{ \quad \{ \quad \phi(x,0) \quad \wedge \quad \forall y \; [\; \phi(x,y) \implies \phi(x,y') \;] \; \} \quad \implies \quad \forall y \; \phi(x,y) \quad \} \tag{1}$$

$$\forall x \, \forall z \, \{ \; \{ \; \phi(x,0) \; \wedge \; \forall y \leq z \, [\, \phi(x,y) \implies \phi(x,y') \;] \; \} \implies \forall y \leq z \, \phi(x,y) \, \} \tag{2}$$

Both Δ_0 induction schemes are known to produce an identical set of theorems when they are combined with Q's axioms. In essence, the first induction scheme appeared in the logic textbooks by Hájek-Pudlák [13], Kaye [15] and Krajícek [18]. Equation (2)'s alternate IΣ_0 induction scheme was used in the Wilkie-Paris article [37]. We will henceforth use the terms **Type-1** and **Type-2** to describe these two induction schemes for IΣ_0.

An initial effort to answer the 1981 Paris-Wilkie open question [23] about whether IΣ_0 would satisfy the Second Incompleteness Theorem for cut-free deduction such as Semantic Tableaux and Herbrand deduction, was undertaken by Adamowicz and Zbierski [1, 4]. They presented a positive solution to the Paris-Wilkie open question [23] for the axiom system IΣ_0+Ω_1. Willard [38, 40] subsequently strengthened their result to show that the Type-1 induction scheme for IΣ_0 satisfied the semantic tableaux version of the Second Incompleteness Theorem. On 16 November 2005, we received a very insightful email communication from L.A. Kołodziejczyk. It pointed out the formal difference between the Type-1 and Type-2 induction schemes for

$I\Sigma_0$ and asked whether it would be possible to generalize [40]'s results to $I\Sigma_0$'s Type-2 induction scheme?

A simple answer to Kołodziejczyk's question is that the basic incompleteness results from [40] will generalize for $I\Sigma_0$'s Type-2 induction scheme. However, a full answer to Kołodziejczyk's fascinating question is somewhat more complicated because our research has also discovered a third class of axiomizations for $I\Sigma_0$, called Ax-3 in our companion paper [45], that proves the same set of theorems as its Type-1 and Type-2 axiomizations but which is an anti-threshold for at least the Herbandized version of the Second Incompleteness Theorem. (The formal definition of an "anti-threshold" for the Second Incompleteness is provided in footnote [1] .) The intuitive reason that Ax-3 will prove the same set of theorems as the Type-1 and Type-2 formalizations of $I\Sigma_0$ — while possessing an essentially opposite incompleteness property — is that these three systems will be unable to formally recognize that they prove the same set of theorems (although they actually do generate identical sets of theorems).

This unexpected contrast high-lights the importance of the open question that Paris-Wilkie posed in their initial 1981 paper [23] and which Leszek Kołodziejczyk (after having conversations with Zofia Adamowicz and Konrad Zdanowksi in Warsaw) essentially reformulated in his November 16, 2005 email to us. It is that the axiom system $I\Sigma_0$ so closely straddles the border between the feasible and infeasible that different logically equivalent versions of it will surprisingly support essentially opposing incompleteness properties !

Our discussion of this topic has been divided into two separate parts. Under this division, our companion paper [45] explores some unorthodox-but-valid axiomizations of $I\Sigma_0$ that should be viewed as boundary-case exceptions for at least Herbrandized version of the Second Incompleteness Theorem. In contrast, the current article focuses instead on generalizations of the Second Incompleteness Theorem for the Type-1 and Type-2 versions of $I\Sigma_0$.

Our research has discovered two slightly different methods available for showing that the Type-2 version of $I\Sigma_0$ has similar incompleteness properties as its Type-1 version. One is a revision of the proof techniques that had appeared in our earlier paper [40]. (We will sketch the structure of this technique later in Appendix C.) An alternate method, which is simpler,

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[1] In a context where D is a deduction method, an axiom system α will be called an "anti-threshold" of the Second Incompleteness Theorem relative to D iff there exists a consistent axiom system α^* that contains all α 's formal axioms and which can prove the following statement:

> "There exists no proof of $0 = 1$ when the deduction method D is applied to the axiom system α^*."

One reason this "anti-threshold" construct is useful is that it signals that a conventional Gódel-like diagonalization proof cannot be applied to analyze the properties of either α or α^* (because the latter's ability to prove the above quoted statement shows that a conventional diagonalization proof must certainly be infeasible.)

uses a new technique, called Passive Induction. It did not appear in [40]. It also allows one to simultaneously prove that both the Type-1 and Type-2 versions of $I\Sigma_0$ satisfy the semantic tableaux version of the Second Incompleteness Theorem. It will be the focus of the current paper.

Passive Induction is likely to be viewed as an interesting technique unto itself, quite apart from quite apart from its relevance to the 25-year old Paris-Wilkie open question, explained in Sections 3 and 4.

2 Notation and Structural Remarks

Our formal definition of a semantic tableaux proof is similar to that appearing in Fitting's and Smullyan's textbooks [11, 29]. It is given below in Definition 1. (For the sake of simplifying our presentation, this definition views the bounded quantifier " $\forall v \leq s$ " as a syntactic object different from the unbounded quantifier of " $\forall v$ ". It likewise separates bounded existential quantifier " $\exists v \leq s$ " from the unbounded existential quantifier " $\exists v$ ".)

Definition 1 . A **Semantic Tableaux Proof** of a theorem Φ will consist of a tree whose root is the sentence $\neg \Phi$ and whose all other nodes are either axioms of α or deductions from higher nodes in the proof-tree. Let the notation " $\mathbf{A} \longmapsto \mathbf{B}$ " indicate that $\mathbf{B}$ is a valid deduction when $\mathbf{A}$ is an ancestor of $\mathbf{B}$ in the proof-tree T . In this notation, the deduction rules allowed under our definition of Semantic Tableaux are:

1. $\Theta \wedge \Gamma \longmapsto \Theta$ and $\Theta \wedge \Gamma \longmapsto \Gamma$.
2. $\neg\neg\Theta \longmapsto \Theta$. Other elimination rules for the " $\neg$ " symbol include:
    (a) $\neg(\Theta \vee \Gamma) \longmapsto \neg\Theta \wedge \neg\Gamma$
    (b) $\neg(\Theta \wedge \Gamma) \longmapsto \neg\Theta \vee \neg\Gamma$
    (c) $\neg(\Theta \Rightarrow \Gamma) \longmapsto (\neg\neg\Theta) \wedge \neg\Gamma$
    (d) $\neg\exists v\, \Theta(v) \longmapsto \forall v \neg\Theta(v)$,
    (e) $\neg\forall v\, \Theta(v) \longmapsto \exists v\, \neg\Theta(v)$
    (f) $\neg\exists v \leq t\ \Theta(v) \longmapsto \forall v \leq t\ \neg\Theta(v)$
    (g) $\neg\forall v \leq t\ \Theta(v) \longmapsto \exists v \leq t\ \neg\Theta(v)$.
3. The pair of sibling nodes Θ and Γ may appear in a candidate tree whenever $\Theta \vee \Gamma$ is their common ancestor.

4. Similarly, a pair of sibling nodes $\neg\Theta$ and Γ may appear a candidate tree when $\Theta \Rightarrow \Gamma$ is their common ancestor.

5. $\exists v \, \Theta(v) \longmapsto \Theta(U)$ where U denotes a *newly introduced* pseudo-constant symbol (often also called in literature a newly introduced "parameter symbol"),

6. For any term s, the Rule 5's analog for bounded existential quantifiers is

$$\exists v \leq s \;\; \Theta(v) \qquad \longmapsto \qquad U \leq s \, \wedge \, \Theta(U)$$

   where U again denotes a *newly introduced* parameter symbol.

7. $\forall v \; \Theta(v) \longmapsto \Theta(t)$ where t denotes a term free in Θ.

8. For any term s, the analog of Rule 7 for bounded universal quantifiers is:

$$\forall v \leq s \;\; \Theta(v) \qquad \longmapsto \qquad t \leq s \, \Rightarrow \, \Theta(t)$$

   where t again denotes a term free in Θ.

Define a particular leaf-to-root branch in a candidate tree T to be **Closed** iff it contains both some sentence Λ and its negation $\neg\,\Lambda$. Then following Smullyan's convention [11, 29], a **Semantic Tableaux** proof of Φ is defined to be a tree whose root stores the sentence $\neg\Phi$ and whose every root-to-leaf branch is closed.

More Notation: Let $\text{SemPrf}_\alpha(x,y)$ denote a Δ_0 formula indicating that y is a semantic tableaux proof of the theorem x from the axiom system α . Given a sequence of integers $i_1,\, i_2,\, i_3,\, ...i_m$, we will say a Δ_0 formula $\Phi(g,j,x)$ makes g an encoding of this sequence iff $\Phi(g,j,x)$ is satisfied only when x represents the $j-$th element in the sequence $i_1,\, i_2,\, i_3,\, ...i_m$. We will say g is a **Linear Compressed Encoding** of this sequence if $\text{Log}(g)$ has a magnitude proportional to the size of $\sum_{j=1}^{m} \text{Log}(i_j + 2)$. Buss, Hájek, Paris, Pudlák and Wilkie [8, 13, 37] have all given examples of such Δ_0 encodings, and we will therefore not also do so here. The Linear Compressed Encodings are considered to be the most efficient possible method to do a formal Gödel encoding of a sentence or of a proof. We will therefore focus on studying such encodings in this paper. (It is probably unnecessary for a reader to examine our formal convention for generating Godel numbers, but a 1-page description of it is available in Appendix A.)

The symbol $\perp$ will denote the Gödel number of the sentence $0 = 1$. The *weakest possible definition* of α's Semantic Tableaux Consistency is the following assertion:

$$\forall p \ \neg \ \text{SemPrf}_{\alpha} (\ \perp \ , \ p \ ) \tag{3}$$

The Definition 2 (below) will offer a different definition of a semantic tableaux deduction and therefore (implicitly) also a different notion of semantic tableaux consistency. Our main interest will be, of course, to derive a final result about semantic tableaux consistency using Equation (3)'s formalization of consistency, rather than to employ the more artificial construct that is implied by the Definition 2 (below). However, one nice aspect of Definition 2's construct is that it is a useful *intermediate step* for analyzing Equation (3)'s notion of semantic tableaux consistency.

Definition 2 . Let $\text{Log}(x)$ denote Base-2 Logarithm, with *downwards rounding* to the lowest integer. (We shall assume $\text{Log}(0) = 0.$) Let $\text{Log}(x, k)$ denote $\text{Log}(\text{Log}(\text{Log}...(\text{Log}(x))))$ − where there are k iterations of logarithm. For any fixed constant $K > 1$, the symbol $\text{SemPrf}^{K}_{\alpha} \ (\ x \ , \ y \ , \ z \ )$ will denote a Δ_0 formula indicating that $\text{SemPrf}_{\alpha} \ (\ x \ , \ y)$ is valid **AND** that $y \ \leq \ \text{Log}(\, z \, , \, K \,)$. Often in this paper when K is a fixed constant, we will use the notation $y \ \leq \ \text{Log}^{K}(\, z \,)$ as an abbreviation for $y \ \leq \ \text{Log}(\, z \, , \, K \,)$.

Remark 1 (concerning Definition 2's notation). It was observed in Benett's Ph. D. dissertation that the graph of exponentiation has a Δ_0 encoding [6]. Expanding upon Benett's idea, Lemma 3.1 of [40] has observed how to formally encode the graphs of $\text{Log}(x, k)$ and hence $\text{SemPrf}^{K}_{\alpha}(x, y, z)$ as Δ_0 formulae. Moreover for a fixed constant K, let us say an integer y is $K-$small iff it satisfies the condition

$$\exists \, z \qquad y \ \leq \ \text{Log}(\, z \, , \, K \,) \tag{4}$$

Many articles in the prior literature [1, 4, 9, 16, 22, 24, 25, 26, 27, 28, 30, 34, 35, 36, 37, 38, 40, 42, 44] have noted that the study of $K-$small integers and the analogous variants of Definition 2's small-proof predicate are useful in generalizing the Second Incompleteness Theorem. These generalizations turn out to be easier to derive when one has available a cut-permissive method of deduction, such as Hilbert deduction, than when only a cut-free deductive calculi is available (such as Herbrand deduction or semantic tableaux). This is because the former can take advantage more easily of the theory for analyzing Definable Cuts on account of its permissible use of a Gentzen-style deductive cut methodology. Prior to our current article, the use of $K-$small integers to develop new theorems about cut-free deduction have included Adamowicz-Zbierski's study of $\text{I}\Sigma_0$+Exp [1, 4], Takeuti's and Kołodziejczyk's analysis [16, 32] of Buss's and Ignjatovic's theorems about Bounded Arithmetic [8, 9], Willard's proof that the Type-1 axiomization of $I\Sigma_0$ satisfies the semantic tableaux version of the Second Incompleteness Theorem [38, 40], and some interconnected generalizations of the latter result in [41, 44].

Definition 3 . Let $D(\alpha)$ denote Gödel's famous diagonalization sentence (formally defined below):

> * There is no Semantic Tableaux proof of **this sentence.**

Also, let $D^K(\alpha)$ denote the following $\mathrm{SemPrf}_\alpha^K(x,y,z)$ modification of this sentence:

> ** In a context where one employs the slightly modified " $\mathrm{SemPrf}_\alpha^K(x,y,z)$ " proof-notation, there exists no code (y,z) that "proves" **this sentence.**

It is very easy to formally encode $D^K(\alpha)$ as a Π_1 sentence following the example of the prior literature on diagonalization. Thus, let $\mathrm{Subst}(g,h)$ denote Gödel's classic Δ_0 substitution relation, defined below:

> $\mathrm{Subst}(g,h)$ = The integer g is an encoding of a formula, and h encodes a sentence identical to g, except that all free variables in g are replaced with a mathematical term representing the numeric quantity of g.

Then $D^K(\alpha)$ can be defined as being the Π_1 sentence $\Gamma(\bar{N})$, where $\Gamma(g)$ denotes the formula (5) and $\bar{N}$ denotes $\Gamma(g)$'s Gödel number.

$$\forall y\ \forall z\ \forall h \quad \{ \quad \mathrm{Subst}(\ g\ ,\ h\ ) \quad \Rightarrow \quad \neg\ \mathrm{SemPrf}_\alpha^K\ (\ h\ ,\ y\ ,\ z\ ) \quad \} \tag{5}$$

The sentence $D^K(\alpha)$ defined above will be henceforth called a **Generalized Godel Diagonalization Sentence**. Thus in a context, where $\bar{N}$ denotes (5)'s Gödel number, the formal mathematical definition of $D^K(\alpha)$ is:

$$\forall y\ \forall z\ \forall h \quad \{ \quad \mathrm{Subst}(\ \bar{N}\ ,\ h\ ) \quad \Rightarrow \quad \neg\ \mathrm{SemPrf}_\alpha^K\ (\ h\ ,\ y\ ,\ z\ ) \quad \} \tag{6}$$

In essence, our version of a proof of the Second Incompleteness Theorem in this paper (and in its earlier version [40]) are similar to the classic diagonalization proofs, except that many intermediate steps will use $D^K(\alpha)$ instead of $D(\alpha)$. One would first naturally suspect that there must be some serious penalty for letting the unconventional diagonalizing sentence $D^K(\alpha)$ replace $D(\alpha)$. However, it will turn out that there is no serious disadvantage for this change because it is only our intermediate steps (rather than our final theorems) that will be affected by this change. We need now to mention one theorem from our earlier paper [40] before we turn to our new results:

Theorem 1 *Suppose α is an extension of Q which (for some standard number K) proves the three theorems below. Then α is inconsistent.*

A $\forall p \; \neg \; SemPrf_{\alpha}(\; \perp \; , \; p \;)$

B $\forall g \; \forall h \; \forall h^{*} \;\; \{ \; Subst(g \, , \, h \,) \; \wedge \; Subst(g \, , \, h^{*} \,) \; \} \;\; \Rightarrow \;\; h \; = \; h^{*}$

C $\{ \;\; \exists y \; \exists z \; SemPrf^{K}_{\alpha} \; (\; \ulcorner D^{K}(\alpha) \urcorner \; , \; y, \; z \;) \;\; \} \;\; \Rightarrow \;\; \exists x \;\; SemPrf_{\alpha} \, (\; \perp \; , \; x \;)$

A roughly 1-page formal proof of Theorem 1 can be found in Section 2 of our earlier paper [40] (under the heading of "Theorem 2.3"). There is no point to repeating that proof here. (It is actually the only theorem that we will be using from our earlier work. The remainder of our discussion in this paper will be new, different, mostly stronger and significantly simpler.) Our formal proof in the next section will actually not use Theorem 1 directly. Instead, it will employ a minor variant of it, which is formalized by Theorem 2. This theorem's statement will actually be the same as the prior statement of Theorem 1 except that it introduces a new constant of M into the formalism and replaces the old Condition-C of Theorem 1 with a new condition C*.

Theorem 2 *Suppose α is an extension of Q where for two fixed integers from the standard model of the natural numbers called K and M, the axiom system α can proves the three theorems below. Then α is inconsistent.*

A $\forall p \; \neg \; SemPrf_{\alpha}(\; \perp \; , \; p \;)$

B $\forall g \; \forall h \; \forall h^{*} \;\; \{ \; Subst(g \, , \, h \,) \; \wedge \; Subst(g \, , \, h^{*} \,) \; \} \;\; \Rightarrow \;\; h \; = \; h^{*}$

C* $\{ \exists y > Log^{K} M \;\; \exists z \; SemPrf^{K}_{\alpha} \, (\ulcorner D^{K}(\alpha) \urcorner, y, z) \, \} \;\Rightarrow\; \exists x \; SemPrf_{\alpha}(\perp, x)$

Proof by Contradiction. If Theorem 2 was false then we could construct an axiom system α that satisfies the following two conditions

i. α is consistent.

ii. α can prove the statements A, B and C*, mentioned in Theorem 2's hypothesis.

The next paragraph will show that Assumptions (i) and (ii) imply that α can also prove the statements A, B and C from Theorem 1's hypothesis. However, the latter is impossible because Assumption (i) indicates that α is consistent, and Theorem 1 indicates that *no consistent* α *can prove* all of statements A, B and C. Using this reasoning, our proof-by-contradiction will establish that it is impossible to construct a system that violates Theorem 2's hypothesis.

More precisely, we need to show that (i) and (ii) imply that α can prove the statement C from Theorem 1's hypothesis. The footnote [2] demonstrates that Item (i) implies that the following statement must be valid in the standard model of the natural numbers:

$$\forall\, y \ \leq \ \mathrm{Log}^K M \quad \neg\ \mathrm{SemPrf}^K_\alpha\ (\ \ulcorner D^K(\alpha) \urcorner\ ,\ y,\ M\ ) \tag{7}$$

Moreover, Equation (7) is similar to a Δ_0 formula because its universal quantifier has only a finite range. It is well known that Q can formally prove all valid such Δ_0-like formulae whose quantifiers have finite range. Hence, Q can prove (7)'s statement. Also from Definition 2's formal description of SemPrf^K, it is obvious that Q, as well as any more elaborate axiom system $\alpha \supseteq Q$, can prove Equation (8)'s statement (see footnote [3])

$$\forall y \leq \mathrm{Log}^K M \ \forall z \ \{ \ \mathrm{SemPrf}^K_\alpha(\ulcorner D^K(\alpha)\urcorner, y, z) \ \Rightarrow \ \mathrm{SemPrf}^K_\alpha(\ulcorner D^K(\alpha)\urcorner, y, M) \ \} \tag{8}$$

In turn, this fact implies that Q can prove prove the Statement C from Theorem 1's hypothesis (because the latter is a consequence of the combination of C* and Equations (7) and (8) — all of which are provable from α). Hence, our proof-by-contradiction has reached its desired end because it has shown that α must satisfy Theorem 2's claim because it would otherwise violate Theorem 1's required constraints. (This is because α would otherwise be a consistent axiom system having a capacity for proving simultaneously Theorem 1's statements A, B and C — a possibility precluded by Theorem 1.) □

Further Introductory Comments. Most of the remainder of this paper will introduce the Passive Induction formalism, that has no direct analog in our prior work. When combined with Theorem 2's machinery, Passive Induction will produce a stronger and simpler analysis for corroborating the $\mathrm{I}\Sigma_0$ version of the Second Incompleteness Theorem than was published in [40].

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[2] The point is that Item (i) indicates that α is consistent, and *no consistent* α can be associated with an ordered pair (y, M) that proves the sentence $D^K(\alpha)$ because it would then prove the diagonalization sentence that states *"There is no ordered pair* (y, M) *that proves me"*. Thus Equation (7) must be certainly valid.

[3] The point is that by definition, $\mathrm{SemPrf}^K_\alpha(\ \ulcorner D^K(\alpha) \urcorner\ ,\ y,\ z\ )$ implies the validity of $\mathrm{SemPrf}_\alpha(\ \ulcorner D^K(\alpha) \urcorner\ ,\ y,)$, which turn implies the validity of $\mathrm{SemPrf}^K_\alpha(\ \ulcorner D^K(\alpha) \urcorner\ ,\ y,\ M\ )$ when $y \ \leq \ \mathrm{Log}^K M$.

Before describing Passive Induction, it is helpful to introduce some further notation. There are essentially three different natural definitions of the Tarski-Mostowski-Robinson, axiom system called Q [33]. The symbol Q_0 will denote the version of Q, used in the current paper. It is is formalized by the Π_1 sentences in Equations (9) – (16).

$$0 = 0 \;\wedge\; 1 = 0' \;\wedge\; 2 = 1' \;\wedge\; 0 \leq 0 \;\wedge\; 0' \neq 0 \;\wedge\; \neg\,[\,0' \leq 0\,] \tag{9}$$

$$\forall\, x \;\;(\;\; x + 0 \;=\; x \;\;\wedge\;\; x \cdot 0 \;=\; 0 \;\;\wedge\;\; x \cdot 1 \;=\; x \;\;) \tag{10}$$

$$\forall x \; \forall y \;\;(\;\; x' \;=\; y' \;\;\Longleftrightarrow\;\; x \;=\; y \;\;) \tag{11}$$

$$\forall x \; \forall y \;\;(\;\; x \;\leq\; y \;\;\Longleftrightarrow\;\; (\; x' \leq y \;\vee\; x \;=\; y \;\;) \tag{12}$$

$$\forall x \; \forall y \qquad x \cdot y' \;=\; (x \cdot y) + x \quad\wedge\quad x + y' \;=\; (x + y)' \tag{13}$$

$$\forall x \; \forall y \; \forall z \quad [\; x = y \;\wedge\; y = z \;] \;\Rightarrow\; [\; x = z \;\wedge\; z = x \;] \tag{14}$$

$$\forall x \; \forall y \; \forall z \quad [\; x = y \;\wedge\; y \leq z \;] \;\Rightarrow\; x \leq z \tag{15}$$

$$\forall x \; \forall y \; \forall z \quad [\; x = y \;\wedge\; z \leq y \;] \;\Rightarrow\; z \leq x \tag{16}$$

Also, let Q_1 denote a system that is begotten by taking the union of Q_0 with Equation (17)'s axiom (which declares that multiplication is a total function). Similarly, Q_2 will denote the union of Q_0 with the Equations (17) and (18) (that indicate multiplication and squaring are total functions).

$$\forall x \;\; \forall y \;\; \exists z \quad x * y = z \tag{17}$$

$$\forall x \quad \exists z \;\; x * x = z \tag{18}$$

It is apparent that all three of these definitions of Q can prove the same set of theorems (since the presence of a multiplication symbol in Equations (9) – (16) assures that Q_0 can prove the validity of the statements that multiplication and squaring are total functions — even if they are not formally incorporated as axioms). Thus, the only difference between Q_0, Q_1 and Q_2 are the lengths of the proofs they generate (rather than the final theorems that they produce).

Also for $i = 1$ or 2 and $j = 0$, 1 or 2, let Type-i-j denote a form of the axiom system $I\Sigma_0$ that combines the Type$-i$ version of $I\Sigma_0$'s induction scheme with the Q_j version of the axiom system Q. Our semantic tableaux version of the Second Incompleteness Theorem in the current article will actually apply to all six of the axiom systems of Type-1-0, Type-1-1, Type-1-2, Type-2-0, Type-2-1 and Type-2-2. However before starting our discussion, it is useful to address a question some logicians have asked us concerning which precise versions of $I\Sigma_0$ did the prior results of [40] actually pertain to?

The brief answer is that [40]'s results, *if very narrowly interpreted,* proved that the Type-1-1 version of $I\Sigma_0$ satisfied the semantic tableaux version of the Second Incompleteness Theorem and that Type-1-2 similarily applied to Herbrand deduction. However, there are several easy generalizations of these results for other paradigms. For instance, Appendix B illustrates that it would be easy to generalize [40]'s results for the Type-1-0 versions of $I\Sigma_0$. Also, the Appendix C (which presupposes that the reader has already examined our earlier article [40]) summarizes how [40]'s formalism can be adjusted to verify the semantic tableaux version of the Second Incompleteness Theorem for each of the Types 2-0, 2-1 and 2-2 axiomizations for $I\Sigma_0$ *without the use of the passive induction technique* introduced in the current article.

Both the Appendices B and C are likely to be difficult to peruse quickly if the reader has not already examined our previous paper [40] in considerable detail. They are not vital for an understanding of the current article because its formalisms will be both self-contained and easier to understand (on account of the new concept of passive induction).

Also as the reader examines the remainder of this article, it should be kept in mind that some unorthodox-but-valid axiomizations of $I\Sigma_0$ (formalized in our companion article [45]) are "anti-thresholds" for at least the Herbrandized version of the Second Incompleteness Theorem (using the terminology from Footnote 1 of Section 1.) Thus, the study of $I\Sigma_0$ is central for understanding the precise threshold-point where the Second Incompleteness Theorem does become active. (The intuitive reason that [45]'s "Ax-3" version of $I\Sigma_0$ will prove the same set of theorems as its sundry Type-1 and Type-2 formalizations — while possessing an essentially opposite incompleteness property — is that these systems will be unable to formally recognize that they prove the same set of theorems — although they actually do generate identical sets of theorems.)

3 Passive Induction

A Δ_0 formula will be called **Passive** iff it includes no free variables. For example, if we write a formula using the *formal* notation convention of " $\phi(x,y)$ " but there are actually no free occurrences of x or y or any other variable in ϕ's body, then we will call this formula "passive". Obviously, if " $\phi(x,y)$ " is passive then there is no strong reason why it was necessary to use the variable names "x" and "y" to designate this formula's name — since it could be more tersely denoted as " ϕ ". However from a *strictly legalistic mathematical* perspective, " $\phi(x,y)$ " is a legitimate *although perhaps redundant* name for this formula.

Let us now apply this concept to the main induction formulae that are used by the axiom system $I\Sigma_0$. These two Δ_0 induction schemes were defined by the Equations (1) and (2) of Section 1, and they are rewritten below. They are called the Type-1 and Type-2 inductive schemes for $I\Sigma_0$. Both schemes prove the same set of theorems when they are combined with Q's axiom system.

$$\forall x \quad \{ \quad \{ \ \phi(x,0) \ \ \wedge \ \ \forall y \ [\ \phi(x,y) \implies \phi(x,y') \ ] \ \} \quad \implies \quad \forall y \ \phi(x,y) \ \ \} \tag{19}$$

$$\forall x \, \forall z \, \{ \ \{ \ \phi(x,0) \ \wedge \ \forall y \leq z \, [\, \phi(x,y) \implies \phi(x,y') \ ] \ \} \implies \forall y \leq z \, \phi(x,y) \, \} \tag{20}$$

Using the nomenclature from the preceding two paragraphs, the induction formulae in Equations (19) and (20) will be called **Passive** when their Δ_0 base formula $\phi(x,y)$ is passive. Although this notation is certainly well defined, many readers will initially feel somewhat uncomfortable with “Passive Induction”. This is because it would initially appear that the meaning of Equations (19) and (20) is quite diluted when they discuss a formula ϕ that has no active free variables of x and y. In other words, the following question will naturally arise in the minds of most meticulous logicians:

> $*$ What is the meaning of Equations (19) and (20) when no variable, including x and y, is free in the formula $\phi(x,y)$? Do these two sentences actually have any greater meaning than a simple trivial 1-atom sentence consisting of the Boolean constant of True?

The answer to the preceding question is oddly both “yes” and “no” — depending on which kind of deduction methodology one is applying. The “no” half of $*$'s answer concerns Hilbert deduction, where a passive induction sentence clearly adds no more useful information than a tautology (that technically contains no formally functional information). On the other hand, if one uses a cut-free method of deduction, such as semantic tableaux, Herbrand deduction or the cut-free predicate form of the sequent calculus (which Takeuti [31] has called “ LK-provability”), then it will turn out that *even the passive form of Equations (19) and (20)'s axioms can cause a theorem's formal proof to have its length drop in size by an exponential or greater magnitude!* It will be on account of this change in proof length that we are able to reply to Kołodziejczyk's emailed questions with a positive answer that our $I\Sigma_0$ Incompleteness Theorem will generalize for the Type-2 axiomizations for $I\Sigma_0$.

To appreciate the nature of this formalism, let α now denote an initial axiom system, and α^* denote a second formalism that includes all α 's axioms plus for every sentence Ψ an added axiom of the form:

$$\Psi \quad \vee \quad \neg \, \Psi \tag{21}$$

From either Godel's Completeness Theorem or Gentzen's Cut Elimination Theorem, it is well known that α and α^* prove the same set of theorems. However, it is also well known that proofs under α^* can be super-exponentially shorter than proofs from α under Semantic Tableaux and Herbrand deduction. This is because Equation (21)'s schema allows α^* to essentially simulate a Gentzen-style cut rule. It is well known that this causes Semantic Tableaux proofs from α^* to have essentially lengths comparable to proofs from α using Hilbert deduction.

The remainder of this article will observe that the passive forms of the induction formulae in Equations (19) and (20) have essentially the same property as the also-redundant axiom in Equation (21). That is, neither will broaden the set of theorems that can be proven. However, passive uses of the induction axioms can exponentially shorten the lengths of cut-free tableaux-like proofs, in a manner that is completely analogous to Equation (21)'s well-known shortening effect for proof lengths.

As a result of this shortening effect, we will derive a simpler proof of the $I\Sigma_0$ incompleteness theorem than had appeared in our previous article. Also, our new result will apply to both the Type-1 and Type-2 axiomizations for $I\Sigma_0$, unlike [40]'s formalism which applied only to the former.

Definition 4 . For $i = 1$ or 2 and $j = 0$, 1 or 2, let us recall that Section 2 had used the symbol Type-i-j to denote a form of the axiom system $I\Sigma_0$ that combines the Type$-i$ version of $I\Sigma_0$'s induction scheme with the Q_j version of the axiom system Q. Most of our discussion in the remainder of this article will concern the special versions of the Type-i-j construct where $j = 0$. It is therefore convenient to use the symbols **T-1** and **T-2** as abbreviations for Type-1-0 and Type-2-0. (Our main results in this paper will state that the T-1 and T-2 versions of the axiom system $I\Sigma_0$ satisfy the semantic tableaux version of the Second Incompleteness Theorem. Since for all i and j , the Type-i-j axiomizations either equal T-1 or T-2 or are a superset of one of these two systems, our main result will imply that each of the six versions of the Type-i-j formalism will also satisfy the semantic tableaux version of the Second Incompleteness Theorem.)

Definition 5 . Let us say an axiom system α satisfies the **Tableaux Δ_0 Compression** property if for every Δ_0 sentence Ψ and for every arbitrary sentence Γ, there will exist assuredly a length O($L+G$) semantic tableaux proof of Γ from α whenever α supports proofs of Ψ and of $\Psi \Rightarrow \Gamma$ with respective lengths of L and G .

Lemma 1 *Some examples of formalisms that satisfy Tableaux Δ_0 Compression are:*

i. *Any axiom system* α *that contains Equation (21)'s axiom "* $\Phi \vee \neg \Phi$ *" for every* Δ_0 *sentence* Φ *.*

ii. *Both the T-1 and T-2 axiomizations for* $I\Sigma_0$*.*

iii. *Any extension of the T-1 and T-2 axiomizations for* $I\Sigma_0$*.*

Proof of Assertion (i) : The validity of Assertion (i) is well known. However, we will review Assertion (i)'s proof because Assertion (ii)'s proof will rest on a more complicated version of the same methodology.

We begin by recalling that Definition 1 indicated that the root of a semantic tableaux proof of the theorem Γ must store the sentence $\neg\,\Gamma$ in the tree's root. Also for any Δ_0 formula Ψ, the sentence $\neg\Psi \vee \neg\neg\,\Psi$ is a formal axiom of α under hypothesis (i). Thus, we may insert this axiom directly below the root in our proof tree. Then using Definition 1's Rule 3 (for $\vee$ elimination), we may insert the following two sibling nodes below this node:

$$\neg\,\Psi \tag{22}$$

$$\neg\,\neg\,\Psi \tag{23}$$

Let S_1 denote a closed subtree descending from Equation (22): We will make its formal structure simply simulate the proof of Ψ . Moreover, since Equation (23) stores the sentence $\neg\,\neg\,\Psi$ and the root (above it) stores $\neg\,\Gamma$, we may hang a subtree S_2 descending from Equation (23), that resembles the proof of $\Psi \Rightarrow \Gamma$. Hence since G and L denotes the lengths of these two proofs (according to Definition 5's terminology), it follows that Γ's proof will have an O($G + L$) length. ☐

Proof of Part (ii) : The analysis of both the T-1 and T-2 axiomizations for $\mathrm{I}\Sigma_0$ are similar. We will therefore focus on the latter for simplicity.

Below is written an instance of Equation (20)'s induction axiom where $\neg\,\Psi$ replaces the generic symbol " $\phi(x,y)$ ". (Here $\neg\,\Psi$ is a passive formula because it has no free variables):

$$\forall x\,\forall z\,\{\;\{\;\neg\Psi\;\wedge\;\forall y \leq z\,[\neg\Psi \implies \neg\Psi\;]\;\}\;\implies\;\forall y \leq z\;\;\neg\Psi\,\} \tag{24}$$

Definition 1 had indicated that the root of a semantic tableaux proof of the theorem Γ will store the sentence $\neg\,\Gamma$. Since an axiom can be inserted anywhere in a semantic tableaux proof, we will make Equation (24)'s sentence the root's child. The next two nodes in our semantic tableaux proof will consist of straight-line path consisting of two applications of Definition 1's $\forall$ Elimination Rule. These applications will eliminate both x and z's universal quantifiers in Equation (24) and replace both these variables with the constant of "1". The resulting great grandchild of the root will then be

$$\{ \neg\Psi \ \wedge \ \forall y \leq 1 \, [\neg\Psi \Longrightarrow \neg\Psi \,] \, \} \ \Longrightarrow \ \forall y \leq 1 \ \ \neg\Psi \tag{25}$$

Next we apply Definition 1's $\Rightarrow$ Elimination rule to the node (25) and thus derive via Definition 1's Rule 4 the following pair of sibling nodes:

$$\neg \, \{ \, \neg\Psi \ \wedge \ \forall y \leq 1 \, [\neg\Psi \Longrightarrow \neg\Psi \,] \, \} \tag{26}$$

$$\forall \, y \, \leq \, 1 \quad \neg\Psi \tag{27}$$

There are eight additional reduction steps, and we will omit itemizing them for the sake of brevity. At the end of these eight steps, our proof tree will be split into four branches (with the terminating nodes A and B descending from Equation (26) and with C and D descending from Equation (27)).

A. $(\neg\neg\neg\,\Psi\,) \ \wedge \ (\neg\neg\ \Psi\,)$

B. $\neg\neg\ \Psi$

C. $\neg\,\Psi$

D. $\neg\ (\ 0 \leq \ 1\ )$ (The footnote [4] justifies Items C and D's presence.))

The nice aspect of this semantic tableaux proof is that the branches descending from nodes A and D terminate promptly because they are inherently contradictory (see the footnote [5] for a clarification). On the other hand, the nodes B and C store the same sentences as the nodes in Part (i)'s proof that stored the contents of Equations (22) and (23). Thus we may end Part (ii)'s proof in the same manner as we had ended Part (i)'s simpler proof.

In particular, we shall insert a copy of Ψ's proof below node C (because it stores "$\neg\ \Psi$") and a copy of the proof of $\Psi \ \Rightarrow \ \Gamma$ below node B (because node B stores "$\neg\neg\,\Psi$" and the

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[4] The quantity "0" is selected when applying Definition 1's Rule 8 (for eliminating the "Bounded $\forall$ " quantifiers) to Equation (27)'s sentence. This rule allows one to choose any term one desires for " t ", including a term (such as "0") that forces a quick contradiction, which pleasantly shortens the length of our semantic tableaux proof). In particular, the application of Rule 8 to Equation (27)'s sentence will produce the deduction of " $0 \leq 1 \ \Rightarrow \ \neg\,\Psi$ ". In turn, an application of the $\Rightarrow$ elimination rule to this last sentence will produce a branch in the proof-tree yielding the two sibling nodes stored in Items C and D.

[5] By "terminate promptly" in the footnoted paragraph, it is meant that the proofs descending from nodes A and D have lengths that are unrelated and independent of the quantities L and G , which Definition 5 had indicated were the length of the proofs for the sentences of Ψ and of $\Psi \rightarrow \Gamma$. Thus when we wish to calculate the length of Γ 's proof as a function of L and G , it is the nodes B and C (rather than A and D) whose subtrees have proof-lengths related to L and G .

tree's root had stored " $\neg \, \Gamma$ "). Hence using Definition 5's terminology, where G and L are the lengths of the proofs for Ψ and for $\Psi \Rightarrow \Gamma$, we obtain that Γ 's proof will have an O($G+L$) length. □

Proof of Part (iii) : Essentially the same as Part (ii)'s proof. □

Definition 6 . For arbitrary n, the symbol Υ_n will denote the following Δ_0 sentence that uses O(n) symbols and asserts the existence of an integer v_n that equals 2^{2^n}.

$$\exists \, v_0 \leq 2 \;\; \exists \, v_1 \leq v_0 * v_0 \;\; \exists \, v_2 \leq v_1 * v_1 \;\; ... \;\; \exists \, v_n \leq v_{n-1} * v_{n-1}$$

$$v_0 = 2 \; \wedge \; v_1 = v_0 * v_0 \; \wedge \; v_2 = v_1 * v_1 \; \wedge \; ... \wedge v_n = v_{n-1} * v_{n-1} \tag{28}$$

Lemma 2 *There exists a constant* d *such that the sentence* Υ_n *has a length O(* n^d *) proof from either the T-1 or T-2 axiomization for* $I\Sigma_0$.

Proof Sketch. Let Φ_0 denote another way of writing the sentence Υ_0 , and let Φ_i be defined below:

$$\Phi_i \quad =_{\text{def}} \quad \{ \, \Upsilon_{i-1} \Rightarrow \Upsilon_i \, \} \tag{29}$$

Then Appendix D will offer a quite routine proof of the following statement

> (+) For some fixed constant c, the sentence Φ_i will have a proof with an O(i^c) length from either the T-1 or T-2 axiomization for IΣ_0 .

Moreover, the sentences Υ_0 , Υ_1 Υ_2 ... are all Δ_0 formulae which satisfy Lemma 1's hypothesis. Therefore by applying n iterations of Lemma 1's Part-ii formalism, it easily follows that Υ_n's proof will have a length proportional to the sums of the lengths for the proofs of the theorems Φ_0, Φ_1, Φ_2, ...Φ_n . In other words, its length is proportional to:

$$\sum_{i=0}^{n} i^c \quad = \quad \text{O}(\, n^{c+1} \,) \tag{30}$$

Hence, Equation (30) implies the validity of Lemma 2 when we define the constant d to equal $c+1$. □

Definition 7 . Let Θ denote some Π_1 sentence. For notational simplicity, let us assume that Θ is encoded as $\forall \, x_1, \; \forall \, x_2, \; ... \; \forall \, x_k \quad \phi(x_1, x_2, ... x_k)$ and that none of the free variable appearing in ϕ include $v_1, v_2, ... v_n$ Then Local(n, Θ) will be *informally defined* to be a Δ_0 sentence asserting the statement:

$$\forall x_1 \leq 2^{2^n} \ , \ \forall x_2 \leq 2^{2^n} \ , \ ... \ \forall x_k \leq 2^{2^n} \qquad \phi(x_1, x_2, ...x_k)$$

We have called the above an "informal definition" of Local(n, Θ) because *it is not technically* a Δ_0 sentence (when it uses a term of the type " 2^{2^n} "). The formal logically precise Δ_0 definition for Local(n, Θ), using Equation (28)'s Υ_n notation, is given below. (Note that the variable v_n in Equation (31) represents the quantity 2^{2^n} . However unlike " 2^{2^n} " itself, v_n's implicit representation of this quantity is permitted to appear in a Δ_0 formula !!!)

$$\exists \, v_0 \ \leq \ 2 \ \ \exists \, v_1 \ \leq \ v_0 * v_0 \ \ \exists \, v_2 \ \leq \ v_1 * v_1 \ \ ... \ \ \exists \, v_n \ \leq \ v_{n-1} * v_{n-1}$$

$$\{ \quad v_0 \ = \ 2 \ \wedge \ v_1 \ = \ v_0 * v_0 \ \wedge \ v_2 \ = \ v_1 * v_1 \ \wedge \ ... \wedge v_n \ = \ v_{n-1} * v_{n-1}$$

$$\wedge \ \ [\ \forall x_1 \leq v_n \ , \ \forall x_2 \leq v_n \ , \ ... \ \forall x_k \leq v_n \quad \phi(x_1, x_2, ...x_k) \ \ ] \ \} \tag{31}$$

To appreciate the meaning of Equation (31), it should be noted that Υ_n, (defined by Equation (28)) has an identical structure as (31) — except that Equation (31) contains an added square bracket expression in its third line.

Lemma 3 *Suppose a proof of the* Π_1 *sentence* Θ *has a length of* L *(from either the T-1 or T-2 axiomization for* $I\Sigma_0$ *). Then for some fixed constant d, the proof of Local*(n, Θ) *will have a length proportional to* $O(\ L \ + \ n^d \ )$*.*

Proof Sketch. Our proof of Lemma 3 will use the fact that the sentence Local(n, Θ) has an identical structure as Υ_n, except that Equation (31)'s definition of Local(n, Θ) contains an additional square bracket expression in its third line. More precisely in a context where Θ has a proof of length L, this implies that " $\Upsilon_n \ \Rightarrow$ Local(n, Θ) " has a proof whose length is bounded by O($L \ + \ n$ Log n) (see footnote [6]).

Moreover, Υ_n falls under the scope of Lemma 1's paradigm (because it is a Δ_0 formula which will satisfy Lemma 1's Tableaux Δ_0 Compression property.) Also, Lemma 2 had indicated that for some fixed constant $d \geq 2$, the sentence Υ_n has a proof with length O(n^d). Thus, we may again apply the combination of Definition 4 and Part ii of Lemma 1 to finish our proof. They imply that Local(n, Θ)'s proof-length is proportional to the sum of the lengths of the proofs for Υ_n and for " $\Upsilon_n \ \Rightarrow$ Local(n, Θ) ". Hence, O($L \ + \ n^d$) bounds the length of its proof. □

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[6] The reason an additional tiny quantity of n Log n is needed to be added to the proof-length is merely to take into account the bit-length for encoding the phrase " Υ_n ". Beyond this point, the claim of the footnoted sentence is obviously valid. This is because Θ has a length L proof and it is obviously a stronger statement than Local(n, Θ).

Theorem 3 . (Unlike our prior paper [40] whose Section 6 applied only to $\mathrm{I}\Sigma_0$'s T-1 axiomization, this theorem will apply to both the T-1 or T-2 axiomizations for $\mathrm{I}\Sigma_0$) *Let* Θ *again denote a* Π_1 *sentence, and suppose that it has a semantic tableaux proof of length* L *from T-i. Also for some sentence* Γ, *suppose that there exists a semantic tableaux proof of length* J *from T-i of the theorem " Local*$(n, \Theta) \quad \rightarrow \quad \Gamma$ *" Then for some fixed constant* d, *there will exist a semantic tableaux proof of* Γ *from T-i whose length is bounded by* $O(\; J \; + \; L \; + \; n^d \;)$.

Proof. Since Θ has a proof of length L from T-i, Lemma 3 indicates Local(n,Θ) will certainly have a proof of length O($L \; + \; n^d$) from T-i, for some fixed constant d. Also, the hypothesis of Theorem 3 stated that there was a proof of length J from T-i of the theorem " Local$(n,\Theta) \quad \rightarrow \quad \Gamma$ " . Applying Lemma 1 to these last two upper bounds on proof lengths, we obtain that Γ has a semantic tableaux proof whose length is no greater than O($J \; + \; L \; + \; n^d$). $\Box$

Theorem 3 is clearly a straightforward consequence of Lemmas 1 – 3 combined with Definition 7's Local(N, Θ) formalism. Nevertheless, it is significant because it allows us to almost (not quite) simulate a Gentzen-style deductive cut rule for Π_1 sentences under either the T-1 or T-2 encodings of $\mathrm{I}\Sigma_0$.

4 The $\mathrm{I}\Sigma_0$ Version of the Second Incompleteness Theorem

This section will explain how Passive Induction enables one to prove that *both the T-1 and T-2* axiomizations of $\mathrm{I}\Sigma_0$ satisfy the semantic tableaux version of the Second Incompleteness Theorem. (Aside from producing a simpler proof methodology than had appeared in our earlier paper [40], the proof in this section is significant because [40]'s theorem technically discussed only the properties of T-1.) It is useful to begin our discussion by reviewing why the question about the properties of $\mathrm{I}\Sigma_0$, raised by Paris and Wilkie in [23], has challenged researchers for the last 25 years.

Let $\omega_1(x)$ denote the usual function operator that maps the integer x onto the quantity $x^{Log(x)}$. Also for a fixed integer K, let $\omega_1^K(x)$ denote the functional operation that maps x onto the number $\omega_1(\; \omega_1(\; \omega_1(\; .... \; \omega_1(\; x))))$ — where there are here K iterations of the operation ω_1 .

In this context it is well known that most axiom systems α have the property that:

$*$ If p is a proof of the theorem T from α, then a proof from α that *"p proves T from α "* will assuredly have a Gödel number less that $\omega_1^K(p)$, for some fixed constant K .

The proof that many axiom systems satisfy the Second Incompleteness Theorem often involves some sub-step, where the concerned system is shown to satisfy the condition $*$.

One complicated aspect of IΣ_0 is that it does not recognize the operation of ω_1 (and therefore also ω_1^K as a "total function"). Thus, IΣ_0 *cannot prove:*

$$\forall\, x \quad \exists\, y \qquad \omega_1(x) = y \tag{32}$$

Nevertheless despite the fact that IΣ_0 cannot prove (32), Wilkie-Paris [37] did show that the Second Incompleteness Theorem held in a very strong sense for IΣ_0 . (They demonstrated IΣ_0+Exp is unable to prove the Hilbert-consistency of a system as weak as Q.)

The proof of this result by Wilkie-Paris used at least a remote analog of Equation (32)'s totality condition. (For those familiar with the theory of Definable Cuts, Wilkie-Paris roughly used the fact that IΣ_0 could essentially treat Equation (32) as being valid within the localized range of a Definable Cut. It turns out that this *localized condition* is sufficient for establishing the validity of the Second Incompleteness Theorem for the axiom system IΣ_0 *under Hilbert deduction.*)

However, the Definable Cut methodology is not well suited for generalizing the Second Incompleteness Theorem to IΣ_0 *under semantic tableaux deduction.* This is because of the cut-free nature of the latter deductive method. It precludes IΣ_0 from possessing certain essential speed-up properties when one uses a cut-free deduction methodology.

Our solution to this problem is based upon using Definition 7's Local(n,Θ) construct to fill in the missing gaps. For an arbitrary integer n and for an arbitrary Π_1 sentence Θ , the Δ_0 sentence Local(n,Θ) will state that Θ is valid for integers as large as 2^{2^n} . The point is that double exponentiation is a much faster growing function than ω_1 . It will be roughly what will be needed for producing our generalization of the Second Incompleteness Theorem.

For the sake of clarity, IΣ_0 certainly *cannot prove* that ω_1 or any faster growing function is total *in a global sense.* However when the axiom system IΣ_0 reasons about its own meta-logical powers, it will know that for a *fixed integer* n, the sentence "Local(n,Θ)" will certainly formalize a very rapid growth among integers *within its own local and fixed domain.* It turns out that this property of Local(n,Θ) will be exactly what is needed to prove that the T-1 and T-2 axiomizations of IΣ_0 satisfy the semantic tableaux version of the Second Incompleteness Theorem.

Definition 8 . The symbol " $\underbrace{N}$ ", will denote the **canonical binary representation** of the natural number N . It will be a term of length $O($ Log N $)$ that defines the value of N using only the constant symbols for the numbers 0, 1 and 2. In particular, let $b_0, b_1, ... b_m$ denote a sequence of bits where $N = \sum_{i=0}^{m} b_i \, 2^i$ and $b_m = 1$. Then $\underbrace{N}$ is defined as:

$$(\; b_0 \; + \; 2 \cdot (\; b_1 \; + \; 2 \cdot (\; b_2 \; + \; 2 \cdot (\; ... \; (b_{m-1} \; + \; 2 \cdot b_m \;)))))$$

Henceforth, the symbol $\lceil \Psi \rceil$ will denote the Gödel number of Ψ (as formally defined by Appendix A). Also, $\underbrace{\lceil \Psi \rceil}$ will denote the canonical binary representation of $\lceil \Psi \rceil$.

Definition 9 . The length (or perhaps one should say "bit-length") of a proof p is represented by the quantity $\text{Log}_2(p)$ when p is written in a binary notation. An often followed notation convention is that $|p|$ denotes $\text{Log}_2(p)$. We shall follow this notation convention in the remainder of this paper.

Definition 10 . Let $i = 1$ or 2 . Throughout this paper, the symbol $\text{SemPrf}_i \, (t, p)$ will denote that p is a semantic tableaux proof of the theorem t from the T-i axiomization of $\text{I}\Sigma_0$. (When we speak of a theorem t possessing a "semantic tableaux proof", we will be referring to this notion, rather than to Definition 2's special artificial concept of a $\text{SemPrf}_i^K(t, p, z)$ construction.)

Definition 11 . A Π_1 sentence $\mho$ will satisfy the **Trivial Manipulation Property** iff it satisfies the following two conditions:

A. $\mho$ is provable from the axiom system $\text{I}\Sigma_0$.

B. In the context of Definition 10's special "$\text{SemPrf}_i \, (t, p)$" notation, an arbitrary Π_1 sentence Ψ will have a proof p satisfying the condition " $\text{SemPrf}_i \, (\; \underbrace{\lceil \Psi \rceil} \; , \; p \;)$ " *only when* there exists a proof q (from the union of the T-i axiomization of $\text{I}\Sigma_0$ with the additional sentence of Local(p , $\mho$)) of the theorem of " $\text{SemPrf}_i \, (\; \underbrace{\lceil \Psi \rceil} \; , \; \underbrace{p} \;)$ " **such that q 's size is governed by the inequality of:**

$$q \; \leq \; \text{O}(\; 2^{2^{\sqrt{p}}} \;) \tag{33}$$

Lemma 4 *There exists a* Π_1 *sentence* $\mho$ *that satisfies Definition 11's Trivial Manipulation Property.*

Informal Justification: Lemma 4 is one of those propositions whose proof is trivial, although somewhat tedious. This is because its formal proof employs a rather lengthy coding argument. We will therefore provide only an intuitive justification of Lemma 4.

The underlying intuition is that Lemma 4 permits one to load any finite number of Π_1 clauses into $\mho$'s formal definition, as long as all these clauses are provable from $\mathrm{I}\Sigma_0$. (This is why Section 2's starting choice of a particular encoding of Q is mostly unimportant, since we can add other clauses later into $\mho$'s formal definition.) Thus loaded into $\mho$ will certainly be the associative, commutative and distributive principles for addition and multiplication. We can even insert axioms declaring that multiplication and squaring are total functions, *provided that they are encoded as* Π_1 *sentences* (rather than in their more abbreviated and conventional Π_2 formats). By this we mean, the formalized Π_1 encodings of these two assertions should resemble the Equations (34) and (35) (whose bound on z's size causes these sentences to be Π_1 statements).

$$\forall x \;\; \forall y \;\; \exists z \; \leq \; x * y \quad x \cdot y \; = \; z \tag{34}$$

$$\forall x \quad \exists z \; \leq \; x * x \quad x \cdot x \; = \; z \tag{35}$$

..... And the list of Π_1 sentences that are available to profitably insert into the sentence $\mho$ goes on ... and on ... and on ... Any finite number of Π_1 sentences that are provable by $\mathrm{I}\Sigma_0$ may be inserted into the sentence $\mho$. This endows it with an enormous amount of potential strength.

The point is that if we add enough sentences to $\mho$'s defining structure, it is easy to establish that if p is an initial proof of some Π_1 theorem Ψ , then $\mathrm{Local}(p, \mho)$ will be rich enough so that its union with the T-i axiom system supports a second proof q, verifying that *"p is a formal proof"* and satisfying Equation (33)'s inequality. The underlying intuition as to why Equation (33) can be satisfied is that nearly all axiom systems can accompany an initial proof p with a second proof q that verifies *" p encodes a proof"* and simultaneously satisfies $q \; < \; \mathrm{O}[\; \omega_1^K(p) \;]$ (for some fixed constant K). Our task is actually much easier than this objective because we need merely verify that q satisfies the *more relaxed* upper bound of $q \; < \; \mathrm{O}[\; 2^{2^{\sqrt{p}}} \;]$. Thus, it is quite easy to at least intuitively appreciate why Lemma 4 must be valid. □

Remark 2 For the sake of brevity, the proof-sketch (above) had not provided a formal description of $\mho$'s required set of Π_1 sentences. However, it is actually easy to provide two additional explanations as to why it is actually quite easy to construct $\mho$. They are given below:

1. It can be formally proven that Equation (33)'s inequality " $q \leq \mathrm{O}(\ 2^{2^{\sqrt{p}}}\ )$ " is a *sharp over-shoot.* That is, one can actually construct a sentence $\mho$ that will force q to satisfy the much tighter constraint of $q \leq \mathrm{O}(\ 2^{\sqrt{p}}\ )$. The reason that we had decided to work with Equation (33)'s more relaxed inequality is merely that it makes it easier to accomplish our main goals and purposes. This is because Equation (33) is such a whopping large inequality and over-shoot that it is trivially obvious that an adequately long list of Π_1 sentences, inserted into $\mho$'s definition, will make this inequality hold.
2. Moreover from a technical standpoint, we do not actually formally need Equation (33)'s inequality of " $q \leq \mathrm{O}(\ 2^{2^{\sqrt{p}}}\ )$ " to formally hold either. This is because the final results of this section will work equally well if we had replaced its inequality's term of $2^{2^{\sqrt{p}}}$ with $2^{2^{2^{\sqrt{p}}}}$ or $2^{2^{2^{2^{\sqrt{p}}}}}$ or with ..... The reason any finite sequence of powers of 2 is adequate on the right side of Equation (33) is that the underlying diagonalization machinery of Theorem 2 is quite broadly encompassing. Thus, if one simply adjusts the exponent K in this theorem to equal the length of the tower of powers of 2, then it will be suitable for justifying the $\mathrm{I}\Sigma_0$ incompleteness theorem, which shall be the final result of this article.

Hence, the point of Items 1 and 2 is that the reader should truthfully not pay too much attention to the details of Equation (33)'s inequality of " $q \leq \mathrm{O}(\ 2^{2^{\sqrt{p}}}\ )$ ". This is because our formalism, based on applying the machineries of Theorems 1 and 2, easily has access to enough degrees of freedom for an entire family of alternative inequalities to be available, if they were needed.

Definition 12 . Let us recall that Benett's dissertation [6] had established that the relation " $y = \mathrm{Log}(x)$ " had a Δ_0 graph, thereby implying that so does automatically " $w = \mathrm{LogLog}(z)$ " have a Δ_0 graph. Using this graph's Δ_0 notation, a Π_1 sentence $\mho^*$ will be said to satisfy the **Trivial Exponentiation Property** iff it satisfies the following two conditions:

A. $\mho^*$ is provable from the axiom system $\mathrm{I}\Sigma_0$.

B. For some fixed constant d and any integer j, the union of the sentence Local(j , $\mho^*$) with either the T-1 of T-2 axiomizations for $\mathrm{I}\Sigma_0$ can support a semantic tableaux proof of $\mathrm{O}(\ j^d\ )$ length proof of:

$$\exists z \qquad \text{LogLog}(z) \;\geq\; \underbrace{j} \tag{36}$$

Lemma 5 *There exists a* Π_1 *sentence* $\mho^*$ *that satisfies Definition 12's Trivial Exponentiation Property.*

Proof Sketch: It was already established by Lemma 2 that there exists a constant d such that the sentence Υ_j has a length O(j^d) proof from either the T-1 or T-2 axiomization for $\text{I}\Sigma_0$. By definition, the $j-$th variable v_j of the sentence Υ_j satisfies the equality

$$\text{LogLog}(v_j) \quad = \quad j \tag{37}$$

Thus, Lemma 2 has already enabled our current proof to be almost done. One Π_1 clause that needs to be inserted into $\mho^*$'s sentence is the statement that: "$\forall\ x\ \text{LogLog}(x^2) \ = \ \text{LogLog}(x) + 1$". (Since the j variables in Υ_j have the property that $v_i \ = \ (v_{i-1})^2$, this equality implies the validity of Equation (37)'s identity.) There is some minor additional work for a formal proof to verify (internally within itself) that the canonical binary term " $\underbrace{j}$ " in Equation (36) actually does formally represent the quantity j from Equation (37). (For instance from a technical standpoint, a proof-formalism does not need any additional Π_1 sentences to formally recognize that " $\underbrace{j}$ " encodes the integer j's value. However, the exponent d for Lemma 5's O(j^d) asymptote can be substantially reduced — if a few additional Π_1 clauses are inserted into $\mho^*$'s structure.) □

Definition 13 . Let $\text{Subst}(g, h)$ denote Gödel's classic Δ_0 substitution relation, whose formal definition is provided below:

> $\text{Subst}(g, h) \quad = \quad$ The integer g is an encoding of a formula, and h encodes a sentence identical to g, except that all free variables in g are replaced with terms using Definition 8's "canonical encoding notation" for defining g's value. (This means that all the free variables in g will be replaced by the term " $\underbrace{g}$ " .)

A Π_1 sentence $\mho^{**}$ will satisfy the **Trivial Substitution Property** iff the following two conditions hold:

A. $\mho^{**}$ is provable from the axiom system $\text{I}\Sigma_0$.

B. Let g and h denote a pair of integers satisfying the condition $\text{Subst}(g, h)$. Then for some fixed integer d, the Δ_0 sentence Local(h , $\mho^{**}$) will support an O(h^d) length proof of the statement "Subst($\underbrace{g}$, $\underbrace{h}$) ", as well as of the statement that h is the unique integer solution for this "Subst" predicate.

Lemma 6 *There exists a* Π_1 *sentence* $\mho^{**}$ *that satisfies Definition 13's Trivial Substitution Property.*

Proof. Trivial and similar in overall structure to the proofs for Lemmas 4 and 5.

Definition 14 . The sentence **Trivial-M** will henceforth denote the conjunction of the three Π_1 sentences of $\mho$ and $\mho^*$ and $\mho^{**}$.

Corollary 1 *The sentence Trivial-M is provable from* $I\Sigma_0$. *It also satisfies the Trivial-Manipulation, Exponentiation and Substitution conditions, formalized by the preceding three lemmas.*

Proof. Since Trivial-M is defined to be the conjunction of the three Π_1 sentences of $\mho$ and $\mho^*$ and $\mho^{**}$, Corollary 1 is an immediate consequence of Lemmas 4 – 6. $\square$

Definition 15 . Let us recall that the "Generalized" Gödel Sentence" $D^K(\alpha)$ was defined by Equation (6). Henceforth the symbols $D^K(1)$ and $D^K(2)$ will denote the particular forms of the $D^K(\alpha)$ diagonalization sentence where α respectively denotes the T-1 and T-2 axiomizations for $\text{I}\Sigma_0$.

Lemma 7 *For the cases where* $i = 1$ *or* 2 *, suppose that* p *satisfies the condition* $SemPrf_i\,(D^2(i) , p)$ *(i.e. it is a semantic tableaux proof of* $D^2(i)$ *from the T-i axiomization for* $I\Sigma_0$ *). Also, let* α_i *denote the union of the T-i with the added sentence of Local(* p *, Trivial-M). Then there will exist a semantic tableaux proof* q *of the sentence* $\neg\ D^2(i)$ *from axiom system* α_i *whose proof length* $|q|$ *is bounded by* $O(2^{\sqrt{p}} + p^d)$ *for some constant* d *(whose value is independent of* p *). More formally, this proof* q *will satisfy the condition:*

$$SemPrf_{\alpha_i}(\neg\ D^2(i) , q) \tag{38}$$

Proof: Lemma 7 is essentially a routine consequence of the combination of Lemmas 1, 2, 4, 5 and 6. We will sketch its proof in substantial detail because it is very central for proving our two main theorems.

Let β_i denote an axiom system that is the union of the axiom system α_i with the added sentence Υ_p (from Definition 6). In a context where q_1 is a proof of Υ_p from α_i and q_2 is a proof of $\neg\ D^2(i)$ from β_i , Part-iii of Lemma 1 implies our proof q can be given a bit-length whose magnitude is proportional to the sum of the lengths of the two proofs q_1 and q_2 . Lemma 2 implies that the former proof will have an O(p^d) length (where d is some fixed constant). Hence to establish, Lemma 7, all we need do is to show that $|q_2|$ can be endowed with an O($2^{\sqrt{p}} + p^d$) bit-length, for at least some constructed fixed constant d.

This is easy to do. Since q_2 formalizes a semantic tableaux proof of the theorem $\neg\, D^2(\ i\ )$, it can be endowed with a proof-tree whose top portion will consist of the following 5 parts:

1. The root of the proof of $\neg\, D^2(\ i\ )$ will contain the usual starting contradiction assumption of $\neg\ \neg\, D^2(\ i\ )$. Applying Definition 1's Rule 2 (i.e. $\neg$ elimination), the child of this root sentence will then be the simple sentence of:

$$D^2(\ i\ ) \tag{39}$$

2. Below Equation (39)'s node will be stored β_i's axiom of Υ_p .

3. A sequence of $p{+}1$ iterations of Definition 1's bounded-existential quantifier elimination rule will then follow. (This is Definition 1's reduction rule 6.) Its $i-$th iteration will eliminate the i-th existential quantified variable v_i from Υ_p and replace it with a parameter U_i (that represents the number of 2^{2^i}).

4. The Definitions 2 and 3 of Section 2 had defined $D^K(\alpha)$ and $\text{SemPrf}^K_\alpha(x,y,z)$. Item 1 of our current construction also has noted that the child of the proof-tree's root is the sentence $D^2(i)$. Unpacking this formal notation into its component parts, the Gödel diagonalization sentence $D^2(i)$ can thus be formally written as:

$$\forall y\,\forall z\,\forall h\ \ \{\ \text{Subst}(\underbrace{N}\,,\,h\,)\ \Rightarrow\ \neg\,[\,\text{SemPrf}^{\,K}_{\,i}\,(h,y)\ \wedge\ y\ \le\ LogLog(z)] \tag{40}$$

in a context where N was the Gödel number of

$$\forall y\,\forall z\,\forall h\ \ \{\ \text{Subst}(\,g\,,\,h\,)\ \Rightarrow\ \neg\,[\,\text{SemPrf}^{\,K}_{\,i}\,(h,y)\ \wedge\ y\ \le\ LogLog(z)] \tag{41}$$

Let M be the Gödel number of Equation (40)'s sentence. Then the next step of our semantic tableaux proof will apply three instances of Definition 1's $\forall$ Elimination Rule to Equation (40)'s stored node. It will replace the variables y, z and h with the terms of $\underbrace{p}$, U_p and $\underbrace{M}$, as shown below:

$$\text{Subst}(\underbrace{N}\,,\,\underbrace{M}\,)\,)\ \Rightarrow\ \neg\,[\,\text{SemPrf}^{\,K}_{\,i}\,(\,\underbrace{M}\,,\ \underbrace{p}\ )\ \wedge\ \underbrace{p}\ \le\ LogLog(U_p)] \tag{42}$$

5. Finally by applying Definition 1's three elimination rules for the $\Rightarrow$, $\neg$ and $\vee$ connectives to (42)'s stored node, three separate branches of the proof-tree will be generated descending from the node (42). The terminating sentences in these three branches will be:

$$\neg \ \text{Subst}(\ \underbrace{N} \ , \ \underbrace{M} \ ) \ \underbrace{p} \ ) \tag{43}$$

$$\neg \ \text{SemPrf}^2_i(\ \underbrace{D^2(\ i \ )} \ , \ \underbrace{p} \ ) \tag{44}$$

$$\neg \ \{ \ \underbrace{p} \ \leq \ LogLog(U_p) \ \ \} \tag{45}$$

The significance of the sentences stored in the nodes (43) – (45) is that the Lemmas 4 – 6 had indicated how closed subtrees can be inserted underneath these three nodes whose bit-lengths, at a minimum, satisfy the formal upper bound O($2^{\sqrt{p}}$ + p^d). (This is because one can think of the nodes (43) – (45) as representing the roots of proof trees that verify the negations of their formal statements.) Hence, these last three parts of the proof q_2 have sufficiently short lengths to finish our justification of Lemma 7. □

Important Reminder: During our proof of Theorem 4, we will rely upon Definition 9's notation. It indicated that $|p|$ denotes the length of a proof p . Its bit-length corresponds to the quantity of $\text{Log}_2(p)$.

Theorem 4 . *Let* $i = 1$ *or* 2 *depending on whether we are discussing the T-1 or T-2 axiomizations for* $I\Sigma_0$. *Let* c *and* d *denote two fixed constants (whose values are independent of* p *). Suppose that* p *is a proof of the Gödel sentence* $D^2(i)$. *Then there exists a proof* q *of* $0 = 1$ *from the T-i axiomization of* $I\Sigma_0$ *whose bit-length satisfies the following inequality:*

$$|q| \quad < \quad c \ \cdot \ [\ 2^{\sqrt{p}} \ + \ p^d \ ] \tag{46}$$

Proof. Some notation will be needed to set up Theorem 4's proof. In a context of proofs that employ the T-i axiomizations for $\text{I}\Sigma_0$, let

1. q_1 denote a proof of " Local(p , $D^2(i)$) "
2. q_2 denote a proof of " Local(p , $D^2(i)$) $\Rightarrow$ $0 = 1$ "
3. q_3 denote a proof of " Local(p , Trivial-M) "
4. q_4 denote a proof of "Local(p,Trivial-M) $\Rightarrow$ [Local($p, D^2(i)$) $\Rightarrow$ $0 = 1$]"

Note that Local(p , $D^2(i)$) is a Δ_0 sentence. The combination of Items (1) and (2) with Part-ii of Lemma 1 therefore implies we can construct a proof q of 0=1 from $\mathrm{I}\Sigma_0$'s T-i axiomization that satisfies the inequality:

$$|q| \quad \leq \quad \mathrm{O}(\ |q_1| \ + \ |q_2|\ ) \tag{47}$$

Likewise because Local(p , Trivial-M) is a Δ_0 sentence, a second application of Part-ii of Lemma 1 implies that we can construct q_2 so that it satisfies

$$|q_2| \quad \leq \quad \mathrm{O}(\ |q_3| \ + \ |q_4|\ ) \tag{48}$$

Thus by combining the results from Equations (47) and (48), it follows that q can be constructed so that it satisfies:

$$|q| \quad \leq \quad \mathrm{O}(\ |q_1| \ + \ |q_3| \ + \ |q_4|\ ) \tag{49}$$

Thus to calculate a bound on $|q|$'s size, we need to estimate the lengths for q_1, q_3 and q_4. This is done below:

a. Since the hypothesis of Theorem 4 indicated that p was a proof of $D^2(i)$, we may apply Lemma 3's formalism to construct a proof q_1 that satisfies (50)'s inequality:

$$|q_1| \quad \leq \quad \mathrm{O}(\ |p| \ + \ p^d\ ) \quad = \quad \mathrm{O}(\ p^d\ ) \tag{50}$$

b. It was already established by Corollary 1 that Trivial-M is provable from $\mathrm{I}\Sigma_0$. Therefore, let r denote such a proof of Trivial-M. In this context, we can apply Lemma 3 to construct a proof q_3 that satisfies the inequality:

$$|q_3| \quad \leq \quad \mathrm{O}(\ |r| \ + \ p^d\ ) \tag{51}$$

Moreover, our O-notation in Equation (51) is intended to describe the properties of a constant c whose value is independent of quantities p and d. Since the proof r is a fixed constant whose structure does not depend in any manner upon p and d, we may therefore absorb it into the O-notation's constant c and thus rewrite this equation in the following simpler form:

$$|q_3| \quad \leq \quad \mathrm{O}(\ p^d\ ) \tag{52}$$

c. Lemma 7 showed a semantic tableaux proof of $\neg\ D^2(i)$ from the union of the T-i axiomization of $\mathrm{I}\Sigma_0$ with the added sentence of Local(p , Trivial-M) must have a proof-length no greater than $\mathrm{O}(\ 2^{\sqrt{p}} + p^d\ )$, for some fixed constant d . The same bound must also apply to a proof of the theorem "Local(p,Trivial-M) $\Rightarrow$ [Local($p, D^2(i)$) $\Rightarrow$ $0 = 1$]" from T-i because these two proofs will have lengths that *differ obviously by no more* than an inconsequentially small number of additional lines. (see the footnote [7] for a detailed justification of this last point).

Thus by substituting these three upper bounds on the lengths of q_1, q_2 and q_4's proofs into Equation (49), one verifies the inequality in Equation (46) that Theorem 4 had claimed. □

Corollary 2 . *The result of Theorem 4 also applies to any axiom system* α *that is an extension of the T-1 or T-2 axiomizations for* $I\Sigma_0$. *Thus, let* c *and* d *denote two fixed constants (whose values are independent of* p *). Suppose that* p *is a proof of the Gödel sentence* $D^2(\alpha)$. *Then there exists a proof* q *of* $0 = 1$ *from* α *whose bit-length satisfies the following inequality:*

$$|q| \quad < \quad c \cdot [\ 2^{\sqrt{p}} \ + \ p^d\ ] \tag{53}$$

.

Proof: Same as Theorem 4's proof. □

Remark 3. The next three propositions in this paper will extend the results of Corollary 2 and Theorem 4 to show that the semantic tableaux version of the Second Incompleteness Theorem is valid for *both the* T-1 and T-2 axiomizations for $\text{I}\Sigma_0$. The same result will also apply to a Herbrandized version of $\text{I}\Sigma_0$'s Incompleteness Theorem. However, it is impractical to cover both topics in one paper. Instead of treating the topic of Herbrandization formally, we have inserted an abbreviated Appendix E into this paper that explains the underlying intuition as to why Herbrand deduction behaves so very similarly to semantic tableaux deduction. (The abbreviated discussion in Appendix E is too brief to be a formal proof. Thus, it may be very beneficial for some other author to compose a more formal proof about this subject.) Since Appendix E assumes that the reader has some prior experience with Herbrand deduction

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[7] The claim made by the footnoted sentence is actually quite trivial to verify. A proof from T-i of

$$\text{" Local(} p \text{ , Trivial-M) } \Rightarrow \text{ [Local(} p \text{ , } D^2(i) \text{) } \Rightarrow \text{ } 0 = 1 \text{] "} \qquad (*)$$

will obviously have the same asymptotic length as a proof from the union of T-i with the added axiom of " Local(p , Trivial-M) " of the theorem " Local(p , $D^2(i)$) $\Rightarrow$ $0 = 1$ ".

Moreover from the definition of the latter sentence, it is easy to confirm that the latter's proof will exceed the asymptotic length of the proof of $\neg\ D^2(i)$ by no more than an additive increment of O(p^d) for some fixed constant d. Hence since Lemma 7 had demonstrated that a semantic tableaux proof of $\neg\ D^2(i)$ from the union of the T-i axiomization of $\text{I}\Sigma_0$ with the sentence Local(p , Trivial-M) will be assured to have a length no greater than O($2^{\sqrt{p}}$ + p^d) for some fixed constant d , the same bound must apply to a proof of the statement (∗).

systems, some readers may prefer to omit this appendix during their first pass through this article.

Lemma 8 *Let* α *denote any axiom system that includes all the formal axioms of either T-1 or T-2. Then there will exist two constants,* K *and* M *, such that* α *can prove the following statement about itself:*

$$\{ \; \exists y > Log^K M \;\; \exists z \;\, SemPrf^K_\alpha \, (\, \ulcorner D^K(\alpha) \urcorner \, , \, y, \, z \,) \; \} \;\; \Rightarrow \;\; \exists x \;\, SemPrf_\alpha(\perp, x) \tag{54}$$

Proof. The entire proof of Theorem 4 and its Corollary 2 can be carried out formally within the domain of $I\Sigma_0$. Therefore, the axiom system α mentioned in Lemma 8's hypothesis can formalize these proofs. This implies that α can also prove Equation (54) because (54)'s statement follows from Theorem 4 when we set $K = 2$ and make M large enough. (The footnote [8] provides the formal details about how this generalization is actually very easy.) □

Theorem 5 *Suppose* α *is an axiom system that contains all the axioms of either the T-1 or T-2 axiomization for* $I\Sigma_0$*. Then if* α *can prove a theorem asserting* **"There exists no semantic tableaux proof of 0=1 from** α **"** *then* α *must be automatically inconsistent.*

Proof. If α could prove the bold-face sentence mentioned in Theorem 5 then it would satisfy the first of the three conditions that Theorem 2 had indicated would make it automatically inconsistent (i.e. Theorem 2's condition A). Moreover, it is obvious that α can also prove Theorem 2's condition (B) (simply because α is an extension of the axiom system $I\Sigma_0$). Furthermore, Lemma 8 showed that α also could prove Theorem 2's third required condition (called C*). Hence if α proves a theorem asserting its own semantic tableaux consistency, then Theorem 2 will imply α is inconsistent. □

Theorem 6 *Every consistent extension of either the T-1 or T-2 axiomizations for* $I\Sigma_0$ *is unable to prove a theorem asserting its own semantic tableaux consistency.*

Proof. An immediate consequence of Theorem 5 because Theorem 6 is a logically equivalent statement, simply rewritten in a contrapositive form. □

Finally, we wish to invite the reader to examine some of our other published papers [39, 42, 43, 44, 45] about this subject. These papers had noted that some formal systems are able

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[8] Since by definition $|q| = \text{Log}_2(q)$, Equation (53) easily implies that $q < 2^{2^p}$ when $q > M$ for some fixed constant M. Thus, Equation (54)'s statement follows when one thinks of its variables y and x as corresponding to p and q from Equation (53) and when one sets $K = 2$ and makes M sufficiently large.

to retain an ability to at least partially corroborate their own consistency in a context where they omit the axiom that multiplication is a total function. (One of these systems is even able to prove the theorem that multiplication is a total function, in a context where [45]'s main result *would collapse entirely* if such a statement about multiplication's totality property was merely changed *from being a theorem into becoming an axiom.* In particular, [45]'s system relies upon a third type of axiomization of $I\Sigma_0$ to partially evade the Second Incompleteness Theorem.)

Thus, part of what makes Theorem 6 interesting is the fact that evasions of the Second Incompleteness Theorem are indeed feasible under axiom systems whose paradigms are only slightly weaker than Theorem 6's setting.

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I also thank the anonymous referee for several very useful suggestions, including the suggestion that Appendix C be added to this article.

Appendix A: Summary of Gödel Encoding Method

This appendix describes our formal method for encoding semantic tableaux proofs as Gödel numbers. Our encoding scheme will be roughly analogous to the "natural B-adic" encoding methods of Hájek-Pudlák [13] and Wilkie-Paris [37] — insofar as the number of utilized bits to encode a semantic object will be approximately proportional to the length of such an expression written by hand. This appendix will be somewhat unnecessary because a reader should be able to appreciate the gist of our article without examining a description of our Gödel encoding method. However in a context where Sections 2–4 often allude to the " Gödel encoding" of a syntactic object, it is clearly helpful to summarize our encoding method.

Our scheme for encoding a semantic tableaux proof will use the following 20 language symbols:

1. The standard connective symbols of $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\forall$ and $\exists$.
2. Two pairs of left and two right parenthesis symbols denoted as: $($, $)$ $\underline{(}$ and $\underline{)}$.
3. Three symbols to represent the special constants of "0", "1" and "2".
4. Three function symbols for representing the operations of addition, multiplication and integer-successor.
5. The relation symbols of " $=$ " and " $\leq$ ".
6. The symbol $\hat{V}$ for designating the presence of a basic variable v in a logical sentence.
7. The symbol $\hat{U}$ for designating the presence of a parameter constant u in a logical sentence (which is produced by Definition 1's deduction rules 5 and 6 for eliminating existential quantifiers).

Define a byte to be an unit consisting of six bits. We will think of a proof as either comprising a sequence of bytes (or being an integer written in base 64). Each of the 20 symbols (above) will be given some unique 6-bit code, ranging between 32 and 51. Our method for representing the presence of the i-th variable v_i will be to encode it is as a string of $\lceil log_{32}(i+1) \rceil + 1$ bytes, where the first byte is the "$\hat{V}$" symbol and the remaining bytes encode i as a base-32 number, with the convention that the lead bit in each byte's 6-bit sequence is "0". The same convention will be used to denote the presence of the i-th parameter u_i except its first byte will be the "$\hat{U}$" symbol.

Our notation has employed *two types* of parenthesis symbols because the first pair of parenthesis symbols will have their usual meaning in punctuating a mathematical sentence, whereas the latter pair of symbols $\underline{(}$ and $\underline{)}$ will *separate* the individual sentences in a Semantic Tableaux proof tree. For example, consider a tree which stores 1) the sentence ψ_1 as its root, 2) the sentences ψ_2 and ψ_3 as the root's children, and 3) ψ_4 as the child of ψ_3. There are several possible notation conventions for using the $\underline{(}$ and $\underline{)}$ symbols to encode a Semantic Proof tree. Our encoding convention will be that ψ_i is an "ancestor" of ψ_j *if and only if* the range beginning with the parenthesis to ψ_i's immediate left and continuing to the matching right parenthesis includes ψ_j. The example of our 4-node proof tree would thus be encoded as:

$$\underline{(}\ \ \psi_1\ \ \underline{(}\ \ \psi_2\ \ \underline{)}\ \ \underline{(}\ \ \psi_3\ \ \underline{(}\ \ \psi_4\ \ \underline{)}\ \ \underline{)}\ \ \underline{)} \tag{55}$$

This byte-styled encoding method is approximately analogous to what Wilkie-Paris [37] have called a *natural B-adic* encoding or a similar counterpart in the Hájek-Pudlák textbook

[13]. Such compressed encodings are considered to be more meaningful and efficient than an uncompressed encoding method, using say a Prime Number decomposition scheme [20] (because the latter has an unnecessarily long bit-length). All our theorems will therefore employ such compressed encoding methods.

Appendix B: An Easy Extension of [40]'s Formalism

This appendix employs the notation from Section 2. Thus, Q_0 will denote an axiomization of Q that uses the Π_1 axioms from Equations (9) – (16). Also, Q_1 will again denote a system that is begotten by taking the union of Q_0 with Equation (56)'s axiom, and Q_2 will represent the union of Q_0 with the Equations (56) and (57).

$$\forall x \;\; \forall y \;\; \exists z \quad x * y = z \tag{56}$$

$$\forall x \quad \exists w \quad x * x = z \tag{57}$$

A key point is that although Equations (56) and (57) are Π_2 sentences, there do exist Π_1 sentences that imply their validity. For example, Equations (58) and (59) are such Π_1 sentences (because the range of their variables z and w are bounded).

$$\forall x \;\; \forall y \;\; \exists z \;\leq\; x * y \qquad x * y = z \tag{58}$$

$$\forall x \quad \exists w \;\leq\; x * x \qquad x * x = z \tag{59}$$

One of the results in [40] was that there existed two Π_1 sentences, called V and V^*, such that:

1. No consistent $\alpha \supset Q_1 + V$ can prove a theorem affirming its own semantic tableaux consistency,
2. No consistent $\alpha \supset Q_2 + V^*$ can prove a theorem affirming its own Herbrand consistency.

Now let W denote the conjunction of the Π_1 sentences of V and V^* with the sentences in Equations (58) and (59). Since $Q_0 + W \supset Q_1 + V$ and $Q_0 + W \supset Q_2 + V^*$, it immediately follows that no consistent $\alpha \supset Q_0 + W$ can prove a theorem affirming either its own semantic tableaux consistency or its own Herbrand consistency.

The latter observation is significant because W is a Π_1 sentence (similar to V and V^*). Section 6 of [40] had shown that Π_1 sentences have a very special property, which its Definition 6.1 had called "Hyper-Constructivity". The Π_1 form of the Hyper-Constructive condition is essentially the following property:

$*$ Any sentence Γ will automatically possess an $\mathrm{O}(G+L)$ length semantic tableaux proof from the Type-1-1 axiomization of $\mathrm{I}\Sigma_0$ when there exists a Π_1 sentence Ψ having a length L proof from Type-1-1 and there also exists a length G tableaux proof of $\Psi \Rightarrow \Gamma$

An application of this hyper-constructive property to the Π_1 sentence V was employed by Theorem 6.4 of [40] to show that the semantic tableaux version of the Second Incompleteness Theorem was valid for the Type-1-1 axiomization for $\mathrm{I}\Sigma_0$. (The idea behind Theorem 6.4 was roughly that the super-exponential growth in proof length, that normally occurs when one connects together two cut-free proofs, can be circumvented when V is the connecting sentence because of its associated $\mathrm{O}(G+L)$ linear growth rate. This fact allowed [40] to conclude that the semantic tableaux version of the Second Incompleteness Theorem held for the Type-1-1 axiomization for $\mathrm{I}\Sigma_0$ because V was a Π_1 theorem of $\mathrm{I}\Sigma_0$.)

The point is that W is also a Π_1 theorem of $\mathrm{I}\Sigma_0$. (Hence, it also falls under the scope of the invariant $*$.) Therefore, one can certainly apply Section 6's full formalism also to it. Thus using the same reasoning as before (see footnote [9]), one obtains that the Type-1-0 axiomization for $\mathrm{I}\Sigma_0$ also satisfies the semantic tableaux version of the Second Incompleteness Theorem.

Hence, if one wishes to meticulously separate the definitions of $\mathrm{I}\Sigma_0$ (depending upon whether they use Q_0's instead of Q_1's definition of Q), then both formalisms will have essentially identical properties. In particular, if one simply replaces the Π_1 sentence V in Section 6 of [40] with the Π_1 sentence of W, then the remainder of [40]'s proof of $\mathrm{I}\Sigma_0$'s semantic tableaux incompleteness property for its Type-1-1 axiomizations generalizes readily for Type-1-0.

Appendix C

Our main proof that the Types 2-0, 2-1 and 2-2 axiomizations for $\mathrm{I}\Sigma_0$ do satisfy the semantic tableaux version of the Second Incompleteness Theorem will appear in Section 4 of this article. This proof will employ Section 3's notion of passive induction as a helpful intermediate step.

It is also possible to devise a different type of proof of this fact using the earlier methodologies from [40]. We will sketch such a proof very briefly in this appendix. Our discussion will presuppose that the reader possesses a detailed knowledge of our article [40]. If a reader has

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[9] The point is that because no consistent $\alpha \supset Q_0 + W$ can prove a theorem affirming its own semantic tableaux consistency and because W is a Π_1 theorem of $\mathrm{I}\Sigma_0$, we can use the invariant $*$ *in the identical manner* as had appeared in Section 6 of [40] to obtain this slightly stronger version of the $\mathrm{I}\Sigma_0$ incompleteness result.

not yet examined [40], he will find the new proof in Sections 2–4 to be easier to comprehend than the sharply abbreviated discussion in this appendix.

The Definition 6.1 from [40] had introduced the notion of an axiom system α possessing an **Hyper-Constructive Semantic Proof** of the theorem Υ . This construct had meant that a sentence " Θ_Υ " existed such that

1. α contains the formal axiom " $\Theta_\Upsilon \Rightarrow \Upsilon$ ", and
2. the sentence Θ_Υ is a theorem of α.

We will also call " Θ_Υ " the **Hyper-Constructed Representative** of Υ.

Let us say a Π_1 sentence Υ is **y–focused** iff it can be written in the form $\forall y \; \phi(y)$ where $\phi(y)$ is a Δ_0 formula. Although Section 6 of [40] did not explicitly deal with the detailed distinction between the Type "1-0" "1-1" and "1-2" constructs, it can be viewed as implying that hyper-connectivity is very useful for analyzing each of these three Type-1 axiomization for $I\Sigma_0$ (see footnote [10]). This is because for each of the cases where j equals 0, 1 or 2, the Type-1-j axiomization for $I\Sigma_0$ will satisfy the following two conditions for every $y-$focused Π_1 sentence Υ :

a. The Type-1-j axiomization for $I\Sigma_0$ will assure that each of its Π_1 theorems Υ will have a Hyper-Constructed Representative, denoted as Θ_Υ and satisfying the conditions (1) and (2) above, such that if Type-1-j supports a length N proof of Υ then it will also support an $O(\,N\,)$ length semantic tableaux proof for Θ_Υ .

b. For an arbitrary sentence Ψ and for an arbitrary y-focused Π_1 sentence Υ (which satisfies Item (a)'s paradigm), if Type-1-j supports an $O(\,M\,)$ length semantic tableaux proof for $\Upsilon \Rightarrow \Psi$ and also an $O(\,N\,)$ length proof for Υ , then it will support an $O(\,M+N\,)$ length semantic tableaux proof for Ψ. (The justification of this fact can be found in Section 6 of [40]. It is essentially a consequence of the combination of Condition (a) and of Part-1 of the definition of a Hyper-Constructed Representative.)

The above two conditions are interesting because semantic tableaux is a cut-free deduction method. Thus, Ψ (in Item b) could potentially have a proof, super-exponentially longer than $M+N$. Items (a) and (b) indicate this problematic exponential growth will simply not occur when the Hyper-Constructive condition is present.

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[10] In particular, the easy argument given in Appendix B explains intuitively why each of Types 1-0, 1-1 and 1-2 have similar properties.

In particular, it was as a result of Item (b)'s linear compression in proof length that Section 6 of [40] was able to establish that the Type-1 axiomizations of IΣ_0 satisfied the semantic tableaux version of the Second Incompleteness Theorem. (Thus, [40]'s proof of this fact rested on essentially combining the prior appendix's methodologies with Item (b)'s linear property for Hyper-Constructivity.)

The remainder of this appendix will explain how an analogous construction can be applied to the Type-2-1 and Type-2-0 axiomizations for IΣ_0. (We do not discuss the Type-2-2 axiomization explicitly in this appendix because it is plainly simpler than the Type 2-1 case.) The proof for each of these cases will be somewhat more complicated than the Type-1 proofs because a precise analog of Hyper-Constructivity will be simply unavailable (since an analog of Condition (a) is absent). However, an analog of Item (b)'s invariant will be established by the combination of the Facts 1 and 2 below:

Fact 1 *Let* Υ *denote a* $y-$*focused* Π_1 *sentence of the form "* $\forall y \; \phi(y)$ *" where* $\phi(y)$ *is* Δ_0*. Suppose that* Υ *has a semantic tableaux proof from the Type-2-1 axiomization of* $I\Sigma_0$ *consisting of* N *sentences. Also suppose that for some term* t *and sentence* Ψ *there exists a proof from the Type-2-1 axiomization of* $I\Sigma_0$ *consisting of* M *sentences of the following theorem:*

$$\phi(t) \quad \Rightarrow \quad \Psi \tag{60}$$

Then there exists a proof from the Type-2-1 axiomization of $I\Sigma_0$ *of* Ψ *which consists of no more than* $O(\,N\,+\,M\,)$ *sentences.*

Proof Sketch. The definition of a semantic tableaux proof can be found in the Fitting's and Smullyan's textbooks [11, 29], as well as in Definition 1 of Section 2. Our proof of Fact 1 will assume the reader is familiar with at least one of these definitions of a semantic tableaux proof. Also, we will use the symbol $\bar{\phi}(y)$ as a convenient abbreviation for the predicate of $\neg\neg\,\phi(y)$. (The footnote [11] explains how $\bar{\phi}(y)$ has different properties from $\phi(y)$ although these two predicates are logically equivalent.)

A semantic tableaux proof of Ψ will begin by storing $\neg\,\Psi$ in its root. The definition of a semantic tableaux proof allows one to store an axiom in any non-root node of this proof. The child of our root will thus store Equation (2)'s Type-2 induction scheme axiom for the predicate $\bar{\phi}(y)$. Its grandchild will store Equation (61)'s sentence. (This grandchild is the

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[11] One of the intermediate steps of the proof-tree that we shall construct will be related to Definition 1's Rule 2-c for eliminating a $\neg$ symbol. Since Rule 2-c produces a double negation term on its right side, the predicate of $\bar{\phi}(y)$ is preferable over the logically equivalent expression of $\phi(y)$.

same as the Type-2 induction axiom except that we have applied the semantic tableaux $\forall$ Elimination Rule to eliminate Equation (2)'s $\forall x$ quantifier. Since the predicate $\bar{\phi}(y)$ does not contain the free variable x in this example, the Equation (61) below does not have any term s operate as a replacement for the so-eliminated variable of x .)

$$\forall z \{ \{ \bar{\phi}(0) \ \wedge \ \forall y \leq z \, [\, \bar{\phi}(y) \implies \bar{\phi}(y') \,] \, \} \implies \forall y \leq z \, \bar{\phi}(y) \, \} \tag{61}$$

We will next apply a second instance of the $\forall$ elimination rule to replace the variable z with the term t in Equation (61). Then the child of Equation (61) will be:

$$\{ \ \bar{\phi}(0) \ \wedge \ \forall y \leq t \, [\, \bar{\phi}(y) \implies \bar{\phi}(y') \,] \, \} \implies \forall y \leq t \, \bar{\phi}(y) \tag{62}$$

A subsequent application of the semantic tableaux $\Rightarrow$ elimination rule will produce the following resultant pair of sibling nodes.

$$\neg \ \{ \ \bar{\phi}(0) \ \wedge \ \forall y \leq t \, [\, \bar{\phi}(y) \implies \bar{\phi}(y') \,] \, \} \tag{63}$$

$$\forall y \leq t \quad \bar{\phi}(y) \tag{64}$$

The remainder of our semantic proof of Ψ will possess a structure similar to its analogs in Section 6 of [40]. Thus a closed subtree with $O(\ N \ )$ sentences can be placed below Equation (63)'s sentence because this subtree will consist of essentially two copies of Υ 's proof. Similarly because $\neg \ \Psi$ is an ancestor of (64)'s sentence, we may insert roughly a copy of Equation (60)'s proof below Equation (64)'s sentence. This proof will consist of M nodes. Hence the full proof of Ψ will consist of no more than $O(N \ + \ M \,)$ sentences. $\Box$

Fact 2 *Let* Υ *denote a* $y-$*focused* Π_1 *sentence of the form "* $\forall y \ \phi(y)$ *" where* $\phi(y)$ *is* Δ_0*. Suppose that*

1. Υ *has a semantic tableaux proof from the Type-2-1 axiomization of* $I\Sigma_0$ *consisting of* N *sentences.*
2. *For some sentence* Ψ*, there is proof of* $\Upsilon \ \Rightarrow \ \Psi$ *from the Type-2-1 axiomization of* $I\Sigma_0$ *consisting of* M *sentences.*

Then there exists a proof of Ψ *from Type-2-1 which consists of no more than* $O(N \ + \ M \,)$ *sentences.*

Proof Sketch: Let us recall that the Type-2-1 axiomization for $\text{I}\Sigma_0$ contains Equation (65) as an axiom.

$$\forall x \;\; \forall y \;\; \exists z \quad x * y = z \tag{65}$$

Using techniques from our article [40], we may construct a sub-fragment of a semantic tableaux proof tree consisting of $O(K)$ sentences (within a straightline path) establishing the existence of a sequence of parameters $u_0, u_1, u_2, \ldots u_K$ such that $u_0 = 2$ and $u_{i+1} = u_i * u_i$. Hence this fragment of our proof-tree will consist of K applications of Equation (65)'s axiom, used to establish the following identity:

$$u_K \; = \; 2^{2^K} \tag{66}$$

Also, the combination of Conditions (1) and (2) in Fact 2's hypothesis easily implies that there exists some integer K whose magnitude is bounded by $O(N+M)$ such that there exists a proof with no more than $O(K)$ sentences of the theorem: $\phi(\, u_K \,) \;\; \Rightarrow \;\; \Psi$. (We omit the formal proof of this fact because it uses reasoning similar to that employed in Section 4. It is also fairly obvious because u_K represents the quantity 2^{2^K} .)

The central point is that the combination of Fact 1 with the observations that $K \; \leq \; O(N+M)$ and that there is a length $O(K)$ proof of $\phi(\, u_k \,) \;\; \Rightarrow \;\; \Psi$ imply Fact 2's validity. $\Box$

The reason Fact 2 is significant is that it establishes an $O(N+M)$ bound on the length of Ψ's proof, similar to Item (b)'s linearity constraint for the Type-1 axiomizations for $\text{I}\Sigma_0$. Thus, the remainder of the justification that the Type-2-1 axiomization for $\text{I}\Sigma_0$ satisfies the semantic tableaux version of the Second Incompleteness Theorem is similar to its analog in Section 6 of our prior paper [40].

We will now turn our attention to the somewhat harder Type-2-0 axiomization of $\text{I}\Sigma_0$. Unlike the Type-2-1 system, it does not contain Equation (65) as an axiom. Therefore it, needs a slightly more complicated argument to establish that its analog of Fact 2 is true. The formalisms of Definition 16 and Fact 3 will enable us to do this.

Definition 16 . Let $\Psi_1, \; \Psi_2, \; \ldots \; \Psi_K$ denote a sequence of sentences that appears within a semantic tableaux proof tree T that employs the axiom system α. A **Semantic Tableaux Fragment** supporting this set of sentences will be defined to be a sub-tree S within the proof-tree T that meets the following three conditions

1. The root of S is some axiom of α.
2. One branch of S , called its **Major Branch**, will contain each of the sentences of $\Psi_1, \; \Psi_2, \; \ldots \; \Psi_K$.

3. All the other branches of S will satisfy the usual semantic tableaux requirement that they contain a pair of contradictory sentences.

Clarifying Comment: We call the reader's attention to the fact that Definition 16 *does not require* the "major branch" to contain a pair of contradicting sentences physically stored within the sub-tree S . (It merely requires that all the other branches of S have this property. The intuitive reason that the major branch is not required to contain a pair of contradictory sentences is that S will be embedded inside a larger proof tree T where every branch does indeed end with a contradiction.)

Fact 3 *Let* u_0 *be another way of writing the named constant of "2" (which was defined by the axiom in Equation (9)). Let* $u_1, u_2, \ldots u_K$ *denote* K *parameters that are defined by Equation (67)'s identity (which causes* $u_i = 2^{2^i}$ *).*

$$u_i \;=\; u_{i-1} * u_{i-1} \;\wedge\; u_{i-1} \;\geq\; 2 \;\wedge\; u_i \;\geq\; 2 \qquad (67)$$

Let $\Psi_1,\ \Psi_2,\ \ldots\ \Psi_K$ *denote the versions of Equation (67) where* i *denotes* 1 , 2 , ... K . *Then for some fixed constant* L *(whose value is independent of* K *), it is always possible under the axiom system T-2 to construct a semantic tableaux fragment which consist of no more than* $L \cdot K$ *nodes and whose major branch includes all the sentences of* $\Psi_1,\ \Psi_2,\ \ldots\ \Psi_K$,

Proof Sketch. We will use the principle of induction to prove Fact 3. Thus, we shall assume that there is available a length $L \cdot (K-1)$ semantic tableaux fragment S_{K-1} representing the sentences $\Psi_1,\ \Psi_2,\ \ldots\ \Psi_{K-1}$. We shall extend this part of our semantic tableaux proof to construct a length $L \cdot K$ semantic tableaux fragment S_K representing the sentences $\Psi_1,\ \Psi_2,\ \ldots\ \Psi_K$.

In our discussion, Ξ will denote the node at the bottom of S_{K-1}'s major branch, and $\phi^*(y)$ will denote the following somewhat awkward-looking Δ_0 formula:

$$\exists\, a \,\leq\, y * y \qquad [\, a \,=\, y * y \;\wedge\; y \geq 2 \;\wedge\; a \geq 2\,] \qquad (68)$$

All the additional nodes of the subtree S_K will be inserted directly below Ξ . The first two of these nodes will consist of the Type-2 induction axiom for the formula $\phi^*(y)$ and the formula that results from the preceding when the $\forall\, x$ quantifier is eliminated. The latter is illustrated below. (It is structured identically to the earlier Equation (61) except that we are now using $\phi^*(y)$ rather than $\bar{\phi}(y)$ as the base predicate.)

$$\forall z\, \{\; \{\; \phi^*(0) \;\wedge\; \forall y \leq z\, [\, \phi^*(y) \implies \phi^*(y')\;]\; \} \implies \forall y \leq z\, \phi^*(y)\, \} \qquad (69)$$

We will now apply a second instance of the $\forall$ elimination rule to replace the variable z with the term u_{K-1} in Equation (69). Then the child of Equation (69) will be:

$$\{ \ \phi^*(0) \ \wedge \ \forall y \leq u_{K-1} \, [\, \phi^*(y) \implies \phi^*(y') \ ] \ \} \implies \forall y \leq u_{K-1} \quad \phi^*(y) \tag{70}$$

A subsequent application of the semantic tableaux $\Rightarrow$ elimination rule will produce the following pair of sibling nodes:

$$\neg \ \{ \ \phi^*(0) \ \wedge \ \forall y \leq u_{K-1} \, [\, \phi^*(y) \implies \phi^*(y') \ ] \ \} \tag{71}$$

$$\forall y \leq u_{K-1} \quad \phi^*(y) \tag{72}$$

We will next apply to the $\forall$ Elimination rule to the node (72) so that y is replaced by the new parameter u_{K-1} and thus obtain the sentence:

$$u_{K-1} \leq u_{K-1} \quad \Rightarrow \quad \phi^*(u_{K-1}) \tag{73}$$

Applying the $\Rightarrow$ elimination rule to the node (73), we then obtain the following pair of two sibling nodes:

$$\neg \ u_{K-1} \leq u_{K-1} \tag{74}$$

$$\phi^*(u_{K-1}) \tag{75}$$

Our construction of S_K is now essentially done. This is because Equation (68)'s definition of ϕ^* (together with the fact that a node storing the sentence Ψ_{K-1} is an ancestor of (75)'s node) trivially implies that one can develop a subtree that descends from Equation (75)'s sentence and whose every branch ends with either the sentence Ψ_K or with a pair of contradicting sentences. Moreover from Equation (68)'s definition for ϕ^* , it is also obvious that one can insert subtrees below the two nodes storing Equations (71) and (74) that contain the required contradictions for closing these branches of the proof tree. Since it is easy to check that this construction assures that the difference between the number of sentences stored in S_{K-1} and S_K is a fixed constant L (whose value is independent of K), our proof of Fact 3 is done. $\Box$

On the Significance of Fact 3: The interesting aspect of Fact 3 is that it shows that the Type-2-1 and Type-2-0 axiomizations of $I\Sigma_0$ have similar properties, despite the fact that the latter does not contain Equation (65)'s axiom that multiplication is a total function. This is because Fact 3 showed that even when this axiom is absent, there still will exist a semantic tableaux fragment with $O(K)$ nodes establishing the existence of the quantity 2^{2^K} , represented by the parameter symbol u_K . (The only difference is that the constant hidden in the O-notation is larger in the latter case, i.e. its O-notation uses a larger constant L — whose value once again does not depend on K.)

As a result of this similarity, our proof that the Type-2-1 axiomization of $I\Sigma_0$ satisfies the semantic tableaux version of the Second Incompleteness version generalizes easily also for its Type-2-0 axiomization. We will not provide more details about this topic here because an alternate and seemingly simpler proof of the same result will appear in Sections 3 and 4, using the notion of a passive induction proof as a simplifying step. The reason that we had briefly sketched an alternate type of proof in this appendix is that it connects more closely to the type of methodologies that were used in our earlier paper [40]. (Thus, one can establish the validity of the semantic tableaux version for the classic axiomizations of $I\Sigma_0$ by using either a variant of Hyper-Connectivity as an intermediate step, or by employing the notion of passive induction that is formalized by Sections 3 and 4 of the current article.)

Appendix D: Finishing Lemma 2's proof

In the context of Definition 6's sentence Υ_i , Lemma 2's proof had formally defined Φ_i to be the sentence:

$$\Phi_i \quad =_{\text{def}} \quad \{ \Upsilon_{i-1} \Rightarrow \Upsilon_i \} \tag{76}$$

Moreover, Lemma 2's proof required that we establish the following invariant be true:

> (+) For some fixed constant c, the sentence Φ_i will have a proof with an $O(i^c)$ length from either the T-1 or T-2 axiomization for $I\Sigma_0$.

The formal proof of Equation (76) will consist of the following 5 steps:

1. The root of Φ_i's proof will contain the usual starting contradiction assumption that its negation is true. Then applying Part 2-c of Definition 1's $\neg$ elimination rule, Equation (77) will represent the root's child.

   $$(\neg \neg \Upsilon_{i-1}) \ \wedge \ \neg \Upsilon_i \tag{77}$$

   A further application of the $\wedge$ elimination rule will produce a straight-line path containing the two sentences of:

   $$\neg \neg \Upsilon_{i-1} \tag{78}$$

   $$\neg \Upsilon_i \tag{79}$$

2. The application of i iterations of Definition 1's $\exists$ elimination rule (together with $i+1$ iterations of Definition 1's $\wedge$ elimination rule) to Equation (78)'s sentence will create $i-1$ new parameters called $U_0, U_1, U_2 ... U_{i-1}$ so that a new node on this straight-line path (after simplification) shall store the sentence

$$U_0 \;=\; 2 \;\wedge\; U_1 \;=\; U_0 * U_0 \;\wedge\; U_2 \;=\; U_1 * U_1 \;\wedge\; ... \wedge U_{i-1} \;=\; U_{i-2} * U_{i-2} \qquad (80)$$

3. The application of $i+1$ cycles of Definition 1's $\neg$ elimination rule to Equation (79)'s sentence will cause another new node on this straight-line path to store the sentence:

$$\forall\, v_0 \;\leq\; 2 \;\; \forall\, v_1 \;\leq\; v_0 * v_0 \;\; \forall\, v_2 \;\leq\; v_1 * v_1 \;\; ... \;\; \forall\, v_i \;\leq\; v_{i-1} * v_{i-1}$$

$$\neg\,[\;\; v_0 \;=\; 2 \;\wedge\; v_1 \;=\; v_0 * v_0 \;\wedge\; v_2 \;=\; v_1 * v_1 \;\wedge\; ... \wedge v_i \;=\; v_{i-1} * v_{i-1} \;\;] \qquad (81)$$

4. The fourth step of our proof will apply $i+1$ iterations of Definition 1's rule for eliminating bounded universal quantifiers to the node that stores Equation (81)'s sentence. For $j \;\leq\; i-1$, the variable v_j will be replaced by the parameter U_j , and Equation (82)'s final variable of v_i will be replaced by the term of $U_{i-1} * U_{i-1}$. The resulting node in our straight-line path will then be:

$$[\;\; U_0 \;\leq\; 2 \;\wedge\;\; U_1 \;\leq\; U_0 * U_0 \;\wedge\; ... \; U_{i-1} \;\leq\; U_{i-2} * U_{i-2} \;\wedge\; U_{i-1} * U_{i-1} \;\leq\; U_{i-1} * U_{i-1} \;]$$

$$\Longrightarrow$$

$$\neg\,[\, U_0 \;=\; 2 \wedge\; U_1 \;=\; U_0 * U_0 \;\wedge\; ... \; U_{i-1} \;=\; U_{i-2} * U_{i-2} \wedge U_{i-1} * U_{i-1} \;=\; U_{i-1} * U_{i-1}]$$

$$(82)$$

5. The last part of our proof tree is based on the observation that the sentences stored in Equations (80) and (82) lie in a straight-line path and contradict each other. The formal reason that they contradict each other is because Equation (80) implies that each parameter U_j equals 2^{2^j} — whereas the " $\neg$ " symbol in the second line of Equation (82) indicates that this cannot be the case. Thus on account of this contradiction, it is possible to develop a further proof subtree that descends from the sentences storing these two nodes, all of whose branches are "closed".

We will not provide further details concerning the construction of Equation (76)'s proof because its overall structure should be evident from the above 5-step description. The fundamental point is that if one works out the details of constructing Step 5's proof tree, then (with a little care) one can assure it will have an O(i^c) length, for some fixed constant c as required by the statement $+$ this appendix has sought to prove.

Appendix E

The abbreviated discussion in this appendix will assume that the reader has already seen a formal definition of Herbrand Deduction, in for example [1, 13, 16]. Roughly speaking a Herbrand proof that an axiom system α *is inconsistent* consists of a 4-step process that 1) first writes α's axioms in their prenex normal forms, 2) next transforms these axioms into Skolemized statements, 3) then generates a finite number of instances of these axioms, *where each instance is produced by replacing all the Skolemized statement's free variables* (from universal quantifier declarations) *with terms built from the combination of the Skolem functions and of the language's initial built-in functions*, and 4) finally develops a propositional calculus style proof that these finite set of instances are unsatisfiable (i.e. that they produce a proof of " $\perp$ "). An interesting facet of Herbrand Deduction is that the set of available Skolem functions makes an *enormous difference* in the efficiency (or proof length) of such a construction.

For instance, let $S(x)$ denote the Skolemized operation of squaring the integer x. Since $S(x) = x{*}x$, it is obvious that multiplication has the logical capacity to simulate a Herbrand function that performs the squaring operation. However, it typically does such a simulation *much less efficiently* than the squaring operation.

Thus, if one starts with a constant c that represents the number 2 and wishes to write the number 2^{2^n} then this quantity could be represented *quite efficiently* by performing n iterations of the squaring function, (i.e. as formally in $2^{2^n} = S^n(c)$ where S^n denotes n iterations of S). *However in contrast,* if only the Skolemized multiplication function symbol is available to write the number 2^{2^n}, then *a very inefficient* $2^n - 1$ applications of the multiplication function are required to write the number 2^{2^n} (i.e. as in a product using the normalized multiplicative form of: $2 * 2 * 2 * \ ... \ * 2$).

Since a major part of our proofs for Lemmas 2 – 7 required calculating numbers of the form 2^{2^n} efficiently, we need some methodology to overcome the preceding paragraph's problem about inefficient uses of multiplication (see the footnote [12] for a clarification). The solution is to move away from the notion of Skolemized functions for constructing our Herbrand proof of the falsity sentence $\perp$ and *instead to use an* **"ascending sequence"** *of Skolemized constants.* As we shall see, the latter can produce constructions as efficient as squaring (and much more efficient than a use of solely the multiplication Skolem function).

More Details about the Formal Nature of "Ascending Sequences" of Skolemized Constants: Four examples will help illustrate how Skolemized constants can *very efficiently* replace Skolemized functions. Consider the prenex normal sentence appearing in Equation (83). Since this sentence contains no universal quantifiers, it is natural to view each existentially quantified variable v_i as being a place holder for a Skolemized constant. (In other words, the "Skolemized function notation" is technically inappropriate for Equation (83) because this sentence contains no universal quantifiers. Instead it is preferable to view the variables $v_0, v_1 ... v_n$ as being transformed into what we shall call an "ascending sequence of Skolemized constants".)

$$\exists\, v_0 \;\exists\, v_1 \;\exists\, v_2 \;...\exists\, v_n \quad \psi(v_0, v_1, v_2 \;...\; v_n) \tag{83}$$

Next in a context where t_0 represents the term "2" and where t_i is the term " $v_{i-1} * v_{i-1}$, let us apply a similar Skolemization methodology to the two logically equivalent sentences below.

$$\exists\, v_0 \;\exists\, v_1 \;...\exists\, v_n \quad [\; v_0 \leq t_0 \;\wedge\; v_1 \leq t_1 \;\wedge\; ...\; v_n \leq t_n \;\wedge \phi(v_0, v_1, \;...\; v_n)] \tag{84}$$

$$\exists\, v_0 \leq t_0 \;\exists\, v_1 \leq t_1 \;\; ...\exists\, v_n \leq t_n \quad \phi(v_0, v_1, v_2 \;...\; v_n) \tag{85}$$

Since Equation (84) has the same generic format as Equation (83), its variables $v_0, v_1, v_2 \;...\; v_n$ should also be viewed as place-holders for Skolemized constants. Moreover since equations (84) and (85) are logically equivalent expressions, it is natural for one to adapt a definition of Skolemizations so that the variables $v_0, v_1, v_2 \;...\; v_n$ are also viewed as place-holders for Skolemized constants in Equation (85). Thus, it is very natural for the bounded existential quantifiers to be treated similarly as unbounded existential quantifiers for the purposes of defining Skolemized constants.

Let us now explore the implications of such a treatment of Equation (85)'s *bounded quantifiers.* Consider the possibility that t_i is defined as above and that $\phi(v_0, v_1, v_2 \;...\; v_n)$ is the

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[12] In other words, our discussion is motivated by the fact that squaring is not an available operation under the T-1 and T-2 axiomizations for $I\Sigma_0$. Thus, we need an alternate methodology to simulate its efficiency. This appendix will be essentially devoted to showing how we can solve this problem with an analog of Definition 6's sentence of Υ_n.

sentence appearing in Equation (86).

$$v_0 = 2 \ \wedge \ v_1 \ = \ v_0 * v_0 \ \wedge \ v_2 \ = \ v_1 * v_1 \ \wedge \ ... \wedge \ v_n \ = \ v_{n-1} * v_{n-1} \tag{86}$$

Then if we substitute Equation (86)'s definition of $\phi(v_0, v_1, v_2 \ ... \ v_n)$ into Equation (85)'s formal axiom, the resulting expression corresponds to Definition 6's sentence of Υ_n. This sentence was the focus of much of our mathematical machinery in Sections 3 and 4 of this article.

The latter observation is useful because it illustrates that we do not need *an explicit use of Skolem functions*, growing at least as fast as squaring, for an axiom system α to efficiently construct the quantity 2^{2^n} via a Skolemization system. Indeed, *if* α *contains the sentence of* Υ_n *as an axiom,* then α can use this particular sentence's formalism to *efficiently simulate* the first n iterations of the operation of squaring. (This is because Υ_n formalizes what we call an "ascending sequence of Skolemized constants".)

We are now ready to collect together the informal arguments given in this appendix to explain, at least intuitively, how the results in Sections 3 and 4 can be generalized from semantic tableaux deduction to Herbrand deduction. It firstly should be noted that the sentence Υ_n is not an axiom of either of our two axiomizations for $I\Sigma_0$, (called its T-1 and T-2 encodings). Therefore, some added work is certainly needed to explain how Υ_n can be treated as essentially as a *virtual axiom*, instead.

The main point is that Υ_n is a Δ_0 sentence. Thus, it is permissible for it to *formally appear* as a passive-inductive subformula within the Type-1 and Type-2 induction axioms of $I\Sigma_0$. Thus, the entire mathematical machinery of Sections 3 and 4 can treat Υ_n as perhaps what should be called an "almost" or "virtual" axiom. By this we mean that analogs of Lemmas 1 and 2 will apply to Herbrand deduction so that the following invariant can be established:

> $*$ For $i \ = \ 1$ or 2, if L denotes the length of some Herbrand proof of a theorem Γ from the union of T-i axiomization of $I\Sigma_0$ with Υ_n, then for some fixed constant d the proof of Γ from T-i will have an O($L \ + \ n^d$) length

Moreover after one has established that Υ_n satisfies the statement $*$'s property of having an almost-axiom-like behavior, the other parts of the machineries of Sections 3 and 4 should generalize as well so as to support the Herbrandized analogs of the $I\Sigma_0$ incompleteness theorems.

Our discussion in this appendix was certainly informal. We felt this article would probably be too long if it sought to deal in depth with both semantic tableaux and Herbrand deduction.

However, we hope that at least our intuitive outline has adequately explained some of the underlying intuitions.

What we are trying to suggest in this appendix is that while a proof of the Herbrandized version of the IΣ_0 incompleteness theorem will be certainly quite interesting and very different from our semantic tableaux analysis, we would anticipate that there are some types of proofs of the Herbrandized version of this incompleteness theorem that partially resemble the analysis techniques used in Sections 3 and 4 for semantic tableaux deduction.

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