

A Version of the Second Incompleteness Theorem For Axiom Systems that Recognize Addition But Not Multiplication as a Total Function

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ABSTRACT: Let $A(x, y, z)$ and $M(x, y, z)$ denote predicates indicating $x + y = z$ and $x * y = z$ respectively. Let us say an axiom system α *recognizes* Addition and Multiplication both as **Total Functions** iff it can prove:

$$\forall x \forall y \exists z A(x, y, z) \quad \mathbf{AND} \quad \forall x \forall y \exists z M(x, y, z) \quad (1)$$

We will introduce some new variations of the Second Incompleteness Theorem for axiom systems which recognize Addition as a “total” function but which treat Multiplication as only a 3-way relation. These generalizations of the Second Incompleteness Theorem are interesting because our prior work [30, 32, 34] has explored several types of boundary-case exceptions to the Second Incompleteness Theorem that occur when one weakens the the hypothesis for our main theorems only slightly further.

1 Introduction

The Second Incompleteness Theorem states that sufficiently strong axiom systems are unable to formally verify their own consistency. There has been

an extensive amount of research about how the Second Incompleteness Theorem can be generalized for weak axiom systems in the context of Frege and Hilbert deduction. For instance, Bezboruah-Shepherdson [3] showed how a version of the Second Incompleteness Theorem is valid for an axiom system that Tarski-Mostowski-Robinson [24] called Q. Pudlák [16] proved a more robust version of the Second Incompleteness Theorem, that applied to all extensions of Q and to all versions of its Gödel encoding (including very importantly localized versions on Definable Cuts) involving Frege-Hilbert style deduction. Wilkie-Paris [29] developed several examples of other versions of the Second Incompleteness Theorems involving for example the inability of $I\Sigma_0 + Exp$ to prove the consistency of Q. Solovay [21] observed how one could modify the formalism of Pudlák [16] with some techniques used by Nelson and Wilkie-Paris [11, 29] to obtain the following result:

* (*Solovay's Extension of Pudlák's Generalization of the Second Incompleteness Theorem [16, 21]*) No reasonable consistent axiom system α treating *merely Successor* as a total function (and viewing Addition and Multiplication as 3-way relations) can recognize the assured non-existence of a Frege-Hilbert style proof of $0=1$ employing α 's axioms.

Our research was greatly stimulated by the Theorem * resulting from the joint work of Nelson, Pudlák, Solovay and Wilkie-Paris. It differs from this theorem's formalism mainly by examining a Semantic Tableaux style of a proof rather than a Frege-Hilbert methodology.

It turns out that some results from Theorem * concerning Frege-Hilbert deduction generalize for Semantic Tableaux, but other aspects of it do not generalize. Our research is also relevant to some open questions raised by Paris and Wilkie [14] concerning the characterization of the exact circumstances where the Semantic Tableaux version of the Second Incompleteness Theorem applies to *weak axiom systems*.

There have been a substantial amount of research in recent years [1, 2, 4, 7, 8, 9, 10, 13, 14, 16, 17, 18, 19, 23, 25, 26, 27, 29, 33] into this topic. Our contribution in [30, 32, 34] was the focus on axiom systems α which *drop* Equation (1)'s assumption that Multiplication is a total function. (Most of [30, 32, 34]'s systems regarded only Addition as total and viewed Multiplication as a 3-way relation.) This topic is interesting because significant boundary case exceptions for the Semantic Tableaux version of the Second

Incompleteness Theorem can be found when an axiom system recognizes merely Addition as *formally total*.

One of the strongest types of boundary-case exceptions to the Second Incompleteness was presented in [34]. It introduced a hierarchy of several increasingly strong definitions of Semantic Tableaux consistency and demonstrated that the prior systems of [30, 32] could be improved so that they had a capacity to recognize their self-consistency under a “Level(1)” definition of Semantic Tableaux consistency (rather than using [30, 32]’s weaker “Level-zero” type definition). The theme of [34] was essentially that all the different levels of definition of Semantic Tableaux consistency were logically equivalent to each other from the standpoint of a strong enough axiom systems. However, a weak axiom system is typically unable to recognize the equivalence of these different levels of definition. Using this fact, [34] demonstrated that some axiom systems were able to evade the force of the Second Incompleteness Theorem when the level-number of their associated definition of Semantic Tableaux consistency was made sufficiently small. In essence, our goal in this paper and in our prior work is to attempt to characterize in as much detail as possible the maximum level number where such an evasion of the Second Incompleteness Theorem is feasible.

Section 2 of this paper will review our definition of different levels of Semantic Tableaux consistency. Our chief goal will be to show that the Semantic Tableaux version of the Second Incompleteness Theorem becomes valid at what is called “Level(2+)” for essentially all axiom systems that recognize merely Addition as a total function. This result is significant because there is a very narrow gap between the Level(1) — where [34] showed the Second Incompleteness Theorem can be evaded when Multiplication is treated as a 3-way relation — and the Level(2+) where the Second Incompleteness Theorem takes force.

There is also a second variant of our results (in both their positive and complementary negative forms) that involves a what we call a TabList style of deduction, described later at the end of the next section.

2 Formal Statement of Main Theorems

Some added notation is needed to define our leveled hierarchy and the various new theorems we will present. Define the mapping $F(a_1, a_2 \dots a_j)$ to be

a **Non-Growth** function iff $F(a_1, a_2, \dots, a_j) \leq \text{Maximum}(a_1, a_2, \dots, a_j)$ for all values of a_1, a_2, \dots, a_j . Six examples of non-growth functions are *Integer Subtraction* (where $x - y$ is defined to equal zero when $x \leq y$), *Integer Division* (where $x \div y$ is defined to equal x when $y = 0$, and it equals $\lfloor x/y \rfloor$ otherwise), *Maximum*(x, y), *Logarithm*(x), *Root*(x, y) = $\lceil x^{1/y} \rceil$ and *Count*(x, j) designating the number of “1” bits among x ’s rightmost j bits. The term **U-Grounding Function** will refer to this set of six non-growth functions plus the Growth operations of Addition and $\text{Double}(x) = x + x$.

All our results in this paper will technically be couched in terms of a language that houses function symbols for the eight operations defined above. This notation is technically unnecessary — because a system that uses the combination of Equation (2) (which implies Addition and Doubling are total functions) along with only our first six non-growth U-Grounding operations would have properties similar to the U-Grounding language.

$$\forall x \forall y \exists z \quad x = z - y \tag{2}$$

However, it is much easier to present a short proof of our results if our language does employ two additional function symbols for the operations of Addition and $\text{Double}(x) = x + x$. The virtue of this notation convention is that it allows us to formally employ constant symbols only for the numbers of 0 and 1 and to encode every other integer $N \geq 2$ using no more than $2 \cdot \text{Log}_2 N$ appearances of the function symbols for Addition and Doubling applied to these two input symbols. Such a mathematical term will be henceforth called the **U-Grounded Binary Representation of N** and be denoted as \overline{N} . For instance, 25 can be encoded in a “binary-like” form as: $1 + \text{Double}(\text{Double}(\text{Double}(1 + \text{Double}(1))))$.

The use of logic’s conventional notation about Π_n and Σ_m sentences is technically inappropriate in this paper because the latter notation (with its Multiplication Function symbol) is suitable only for axiom systems which recognize Multiplication as a total function. Instead, our analogs for Π_n and Σ_m in the U-Grounding Function language are called Π_n^* and Σ_m^* . Here, a *term* t is defined to be a constant, variable or a U-Grounding function symbol (whose input arguments are recursively defined terms). Also, the quantifiers in the wffs $\forall v \leq t \Psi(v)$ and $\exists v \leq t \Psi(v)$ are called *bounded quantifiers*. If Φ is a formula that uses the U-Grounding primitives as its function symbols and the two relation symbols of “=” and “ \leq ”, then

this formula will be called both Π_0^* and Σ_0^* whenever all its quantifiers are bounded. For $n \geq 1$, a formula Υ shall be called Π_n^* iff it is written in the form $\forall v_1 \forall v_2 \dots \forall v_k \Phi$, where Φ is Σ_{n-1}^* . Likewise, Υ is called Σ_n^* iff it is written in the form $\exists v_1 \exists v_2 \dots \exists v_k \Phi$, where Φ is Π_{n-1}^* .

Let us call Υ a Q_n^* sentence iff it is one of a Σ_n^* sentence, a Π_n^* sentence or a Boolean combination of several Σ_n^* and Π_n^* sentences (using the standard connective symbols of \wedge , \vee , \neg and \rightarrow). There will be three types of definitions of Semantic Tableaux consistency that we will examine in this paper. They are defined below:

1. A **Level(n) Definition** of an axiom system α 's Tableaux consistency is the declaration that there exists no Π_n^* sentence Υ supporting simultaneous Semantic Tableaux proofs from α of both Υ and its negation.
2. A **Level(n+) Definition** of an axiom system α 's Tableaux consistency is the statement that no Q_n^* sentence Υ supports simultaneous Semantic Tableaux proofs for both Υ and its negation.
3. A **Level(0-) Definition** of a system α 's Tableaux consistency is the statement that there exists no proof of $0=1$ from α .

All definitions of consistency, from Level(0-) up to Level(n+) *for any* n , are equivalent to each other under strong enough models of Arithmetic. However, many weak axiom systems do not have a mathematical strength to formally prove and recognize this equivalence.

Translated into this Π_1^* styled notation convention, the core result in [34] was the construction of a consistent axiom system α which had the following properties:

1. α was capable of recognizing its Level(1) Tableaux consistency.
2. α was capable of recognizing Addition as a total function.
3. α was capable of proving all of Peano Arithmetic's Π_1^* theorems.

The above result beckons one to consider whether or not it would be possible to develop stronger versions of this effect where α could recognize higher levels of its own Tableaux consistency. Theorem 1 and Remark 1 show that most such generalizations are infeasible.

Theorem 1. *Let α denote an axiom system that uses the language of the U-Grounding functions (and thus recognizes Addition as a total function). There exists a Π_1^* theorem W of Peano Arithmetic such that no consistent $\alpha \supset W$ of finite cardinality can recognize its own Level(2+) Tableaux consistency.*

Remark 1. It is also possible to generalize Theorem 1 for essentially all axiom system $\alpha \supset W$ of infinite cardinality. In particular, let us say an axiom system α satisfies the **Conventional Deciphering Property** iff there exists a Σ_0^* sentence $\text{Test}(n)$ such that n represents the Gödel number of an axiom of α iff and only iff $\text{Test}(n)$ is true. We will not have the page space to prove this stronger result here, but Theorem 1 can be strengthened to indicate that no consistent axiom system $\alpha \supset W$ satisfying the Conventional Deciphering Property can prove a theorem affirming its own Level(2+) Tableaux consistency.

Remark 2. The Incompleteness effect described by Theorem 1 and Remark 1 should not be confused with a prior result published by us in [33]. The latter's version of the Incompleteness Theorem showed that there were essentially no interesting axiom systems that could simultaneously recognize their Level(0-) Tableaux consistency and also recognize both *Addition and Multiplication* as total functions. This alternate effect is quite different from our current result, which will apply to axiom systems that recognize *solely Addition* as a total function.

In particular, this distinction is non-trivial essentially because [34] showed that axiom systems could recognize their Level(1) consistency when they treated Multiplication as a 3-way relation (rather than as a total function). Hence, there arises the question concerning at what Level of Tableaux consistency does the Semantic Tableaux version of the Second Incompleteness become valid for this latter class of axiom systems which do not employ [33]'s assumption that Multiplication is a total function? The purpose of Theorem 1's generalization of the Second Incompleteness Theorem is to at least partially answer this question. It shows that while [33]'s Tableaux generalization of the Second Incompleteness Theorem is known from [34] to become false at all levels between 0 – and 1 when a system fails to recognize Multiplication as total, there is nevertheless available a Level(2+) Tableaux generalization of the Second Incompleteness Theorem for such systems.

We will also discuss in this paper a cousin of Theorem 1's Incompleteness

result, involving an alternative rule of inference, called TabList deduction. It is helpful to review Smullyan's formal definition of a Semantic Tableaux proof before formally defining TabList deduction.

Following roughly Fitting's or Smullyan's notation [5, 20], let us define a **Φ -Based Candidate Tree** for the axiom system α to be a tree whose root corresponds to the sentence $\neg\Phi$ and whose all other nodes are either axioms of α or deductions from higher nodes of the tree. Let the notation " $\mathcal{A} \implies \mathcal{B}$ " indicate that \mathcal{B} is a valid deduction when \mathcal{A} is an ancestor of \mathcal{B} . In this notation, the Tableaux-Deduction rules are:

1. $\Upsilon \wedge \Gamma \implies \Upsilon$ and $\Upsilon \wedge \Gamma \implies \Gamma$.
2. $\neg\neg\Upsilon \implies \Upsilon$. Other deduction rules for the \neg symbol include:
 $\neg(\Upsilon \vee \Gamma) \implies \neg\Upsilon \wedge \neg\Gamma$, $\neg(\Upsilon \rightarrow \Gamma) \implies \Upsilon \wedge \neg\Gamma$, $\neg(\Upsilon \wedge \Gamma) \implies \neg\Upsilon \vee \neg\Gamma$,
 $\neg\exists v \Upsilon(v) \implies \forall v \neg\Upsilon(v)$ and $\neg\forall v \Upsilon(v) \implies \exists v \neg\Upsilon(v)$
3. A pair of sibling nodes Υ and Γ is allowed when their ancestor is $\Upsilon \vee \Gamma$.
4. A pair of sibling nodes $\neg\Upsilon$ and Γ is allowed when their ancestor is $\Upsilon \rightarrow \Gamma$.
5. $\exists v \Upsilon(v) \implies \Upsilon(u)$ where u is a newly introduced Parameter Symbol.
6. $\forall v \Upsilon(v) \implies \Upsilon(t)$ where t denotes a "Function Term". These terms are U-Grounding Function objects, whose inputs are any set of constant symbols, parameter symbols or other function-objects.

Define a particular leaf-to-root branch in a candidate tree T to be **Closed** iff it contains both some sentence Υ and its negation $\neg\Upsilon$. A **Semantic Tableaux** proof of Φ is then defined [5, 20] to be a candidate tree whose root stores the sentence $\neg\Phi$ and all of whose root-to-leaf branches are closed.

One further definition is needed before we can describe the second variation of Theorem 1's Incompleteness Result explored in this paper. Let H denote a sequence of ordered pairs $(t_1, p_1), (t_2, p_2), \dots, (t_n, p_n)$, where p_i is a Semantic Tableaux proof of the theorem t_i , and let \mathfrak{R} denote an arbitrary class of sentences. Define H to be a Tab- \mathfrak{R} -List proof of a theorem T from the axiom system α iff $T = t_n$ and also:

1. Each axiom in p_i 's proof is either one of t_1, t_2, \dots, t_{i-1} or comes from α .

2. Each of the “intermediately derived theorems” t_1, t_2, \dots, t_{n-1} must lie within the “prespecified class” \mathfrak{R} of sentences.

If \mathfrak{R} denotes the set of Q_k^* sentences, the notion of an Tab- Q_k^* -List proof is quite similar (although not fully identical) to constructs that have been called R-proofs and Q_k style proofs by Hájek, Paris, Pudlák and Wilkie in [7, 16, 29]. One minor difference between these definitions is that the TabList notion contains some added flexibility because it allows one to set \mathfrak{R} equal to any of the classes of Π_k^* sentences, Σ_k^* sentences, Q_k^* sentences, or for example the union of the sets of Π_k^* and Σ_k^* sentences. Another difference is that the R-proofs and Q_k style proofs of [7, 16, 29] are based on partially limiting the power of Hilbert-style deduction, whereas our dual form of this construct proceeds in the opposite direction — where we seek to progressively expand the logical power of Semantic Tableaux style deduction instead.

For any class \mathfrak{R} of sentences, each of our prior definitions of Level(0), Level(N) and Level(N+) consistency can be generalized for Tab- \mathfrak{R} -List deduction. For instance, an axiom system α 's Level(0-) consistency under Tab- \mathfrak{R} -List deduction is the statement that every Tab- \mathfrak{R} -List proof from α 's axioms fails to prove $0=1$.

Below are our main theorems about the generality and limitations of the Second Incompleteness Theorem under TabList deduction.

Theorem 2. *Let α denote an arbitrary axiom system that uses the language of the U-Grounding functions (and thus recognizes Addition as a total function). It is not necessary, but for the sake of simplifying our proof of Theorem 2 we will also assume that the axiom system α has finite cardinality. Then there exists two Π_1^* theorems of Peano Arithmetic, V_A and V_B such that*

- A. *No consistent $\alpha \supset V_A$ can prove a theorem affirming its own Level(0-) consistency under Tab- Π_2^* -List deduction.*
- B. *No consistent $\alpha \supset V_B$ can prove a theorem affirming its own Level(0-) consistency under Tab- Σ_2^* -List deduction.*

Theorem 3. *Let Tab₁List be an abbreviation for the variant of Tab- \mathfrak{R} -List deduction where \mathfrak{R} denotes the union of the set of Π_1^* and Σ_1^* sentences. Then for each consistent axiom system A that is an extension of Peano Arithmetic, there exists a consistent axiom system α that can*

1. recognize its own *Level(1)* consistency under *Tab₁List* deduction.
2. recognize the validity of all *A*'s Π_1^* theorems, and
3. recognize *Addition* as a total function

Part of the reason Theorems 2 and 3 are interesting is due to the close match between their complementary positive and negative results. Thus, Theorem 3 established that there exists a Boundary-Case exception to the Second Incompleteness Theorem when \mathfrak{R} represents the union of the set of Π_1^* and Σ_1^* sentences, while Theorem 2 shows the Second Incompleteness Theorem comes to force when \mathfrak{R} represents instead either the class of Π_2^* sentences or the class of Σ_2^* sentences. Moreover, Theorem 3 indicates that its Boundary-Case exception rises up to *Level(1)* definitions of consistency, while Theorem 2 shows that even the lower *Level(0-)* is problematic under *Tab- Π_2^* -List* and *Tab- Σ_2^* -List* deduction.

Most of our discussion in this paper will focus on proving Theorems 1 and 2. Theorem 3's result was technically announced on the last page of our *Tableaux-2002* conference paper [34]. However, the latter conference paper was written in a too abbreviated style for it to also include a proof of Theorem 3. Instead, its formal proof examined a slightly more specialized variation of Theorem 3 where *Semantic Tableaux* deduction replaced *Tab₁List* deduction (in Clause 1 of Theorem 3). We have therefore also inserted a 3-page appendix into the current article, which roughly outlines how [34]'s proof formalism can be slightly strengthened to obtain Theorem 3's more general result. This appendix is helpful because there is a pleasantly tight and sharp match between Theorem 3's positive result and Theorem 2's complementary negative result, as was explained in the prior paragraph.

3 Overall structure of Theorem 1's Proof

Our method for encoding a *Semantic Tableaux* proof p is described on page 581 of our article [32]. This encoding is maximally compressed in that it will encode p as an integer whose “**Bit-Length**” is approximately proportional to the length of such a *Semantic Tableaux* proof when it is written down by hand. Several other authors [7, 29] have also employed roughly similar types of maximally compressed encoding methods. It is therefore probably unnecessary for a reader to examine our exact encoding method in [32].

This section will sketch the overall structure of Theorem 1's proof. Let $\text{Prf}_\alpha(x, y)$ denote a Σ_0^* formula indicating y is a semantic tableaux proof of the theorem x from the axiom system α . Also, let $\text{Log}(z)$ denote Base-2 Logarithm, with *downwards rounding* to the lowest integer, and $\text{Log}^\lambda(z)$ denote the operation $\text{Log}(\text{Log}(\text{Log} \dots (\text{Log}(z))))$ — where λ designates an integer indicating the number of iterations of Log here. It is useful to employ the notation $\text{ShortPrf}_\alpha^\lambda(x, y, z)$ to denote a Σ_0^* formula indicating that y represents a Semantic Tableaux proof of the theorem x from the axiom system α **and** that $y = \text{Log}^\lambda(z)$.

Takeuti [23] introduced a form of the λ -Short-Proof concept for studying integers y satisfying the condition $\exists z \text{Log}^\lambda(z) = y$. His goal was to use this construct to help explicate the relationship between Buss's Bounded Arithmetic, Gentzen's sequent calculus and some of NP's properties [23]. An entirely different type of application of the λ -Shortness concept was subsequently observed by Adamowicz, Salehi, Willard and Zbierski [1, 2, 19, 31, 33] (largely independently of Takeuti's research). This second line of research used the λ -Shortness concept as an intermediate step to help answer some open questions about $\text{I}\Sigma_0$'s Incompleteness properties raised by Paris and Wilkie in [14]. Thus in approximate chronological order, the latter research included Adamowicz-Zbierski's observation [1, 2] that a cut-free version of the Second Incompleteness Theorem was valid at the level of $\text{I}\Sigma_0 + \Omega_1$, Willard's strengthening of this result so that the threshold for the Cut-Free Second Incompleteness effect would be lowered so that it would include all extensions of $\text{I}\Sigma_0$ and most extensions of Q [31, 33], and Salehi's more recent second type of proof [19] of Willard's $\text{I}\Sigma_0$ Incompleteness Theorem.

The λ -Short concept will also help us prove Theorem 1 in this paper. Thus, let $D(\alpha)$ denote the following Gödel sentence:

“There is no Semantic Tableaux proof of **this sentence** from α 's set of proper axioms”

Let $D^\lambda(\alpha)$ denote the “ $\text{ShortPrf}_\alpha^\lambda(x, y, z)$ ” analog of this diagonalization sentence, defined below:

“In a context of the $\text{ShortPrf}_\alpha^\lambda(x, y, z)$ notation convention, there exists no code (y, z) that *proves* **this sentence** from α 's axioms”

It is easy to give $D^\lambda(\alpha)$ a Π_1^* encoding. Thus, let $\text{Subst}(g, h)$ denote the following Σ_0^* formula:

Subst(g,h) = The integer g is an encoding of a formula, and h encodes a sentence identical to g , except all g 's free variables are now replaced by a term equal to the constant g itself.

Then following Gödel's example, $D^\lambda(\alpha)$ is formally defined to be the sentence $\Gamma(\bar{n})$, where Equation (3) defines the formula $\Gamma(g)$ and \bar{n} is a term whose numerical value represents $\Gamma(g)$'s Gödel number.

$$\forall y \forall z \forall h < y \quad \{ \text{Subst}(g, h) \rightarrow \neg \text{ShortPrf}_\alpha^\lambda(h, y, z) \} \quad (3)$$

Our proof of Theorem 1 will use the $D^\lambda(\alpha)$ Diagonalization sentence as an intermediate step to help corroborate Theorem 1. In particular, let **Pair(s,t)** denote a Σ_0^* formula indicating that s is the Gödel number of a Q_2^* sentence and that t is the Gödel number of a second Q_2^* sentence which is the negation of s . Also, let $\lceil D^\lambda(\alpha) \rceil$ denote $D^\lambda(\alpha)$'s Gödel number. Then the Theorem 4 (below) will help prove Theorem 1.

Theorem 4. *Suppose α is a consistent axiom system capable of proving all the Σ_0^* sentences that are valid in the Standard Model. Suppose there exists two constants, λ and L , such that α can also prove:*

$$A) \quad \forall g \forall h \forall h^* \quad \{ \lceil \text{Subst}(g, h) \wedge \text{Subst}(g, h^*) \rceil \rightarrow h = h^* \}$$

$$B) \quad \forall z > L \quad \forall y \quad \{ \text{ShortPrf}_\alpha^\lambda(\lceil D^\lambda(\alpha) \rceil, y, z) \rightarrow$$

$$\exists p < z \exists q < z \exists s < z \exists t < z \lceil \text{Pair}(s, t) \wedge \text{Prf}_\alpha(p, s) \wedge \text{Prf}_\alpha(q, t) \rceil \}$$

Then α must be incapable of proving:

$$\forall p \forall q \forall s \forall t \quad \neg \lceil \text{Pair}(s, t) \wedge \text{Prf}_\alpha(p, s) \wedge \text{Prf}_\alpha(q, t) \rceil.$$

We will not prove Theorem 4 here because its justification is similar to the Theorem 2.3 from [33]. The remainder of this article will use Theorem 4 as an intermediate step to help prove Theorems 1 and 2.

4 The Formal Proof of Theorem 1.

Let α denote an axiom system and $\varphi(x)$ denote a formula free in only x . The formula $\varphi(x)$ is called [7] a **Definable Cut relative to α** iff α can prove the theorem:

$$\varphi(0) \quad \text{AND} \quad \forall x \varphi(x) \rightarrow \varphi(x+1) \quad \text{AND} \quad \forall x \forall y < x \varphi(x) \rightarrow \varphi(y) \quad (4)$$

Definable Cuts have been studied by a very extensive literature [1, 2, 4, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 22, 25, 26, 27, 28, 29]. They are unrelated to Gentzen's notion of a Sequent Calculus "Deductive Cut Rule".

It is convenient to call a Definable Cut **trivial relative to α** iff α can formally prove " $\forall x \varphi(x)$ ". For example since Peano Arithmetic recognizes the validity of the Principle of Induction, all its Definable Cut Formulae are trivial. On the other hand, every arithmetical logical system strictly weaker than Peano Arithmetic contains some non-trivial Definable Cut.

One theme of the literature about Definable Cuts is that they are very helpful for developing new versions of the Second Incompleteness Theorem, as well as for devising new uses of it. For instance, the Theorem * , attributed in Section 1 to the joint work of Nelson, Pudlák, Solovay and Wilkie-Paris, was derived by using Definable Cuts as a crucial intermediate step. As we pointed out in Section 1, our new Theorem 1 is related to this literature, and it was greatly stimulated by its over-all perspective. However, one aspect of our main proof will veer in a slightly different direction.

The distinction arises because the Second Incompleteness Theorem is much more difficult to prove and generalize for cut-free proof methods, such as Semantic Tableaux or the Cut-Free variant of the Sequent Calculus, than it is for Cut-Permissive formalisms, such as the Hilbert-style methodology or the Sequent Calculus with a Cut Rule. The intuitive reason for this distinction is that most generalizations of the Second Incompleteness Theorem using Equation (4)'s formalism require a Gentzen-style Deductive Cut Rule (or equivalently some Hilbert-style modus ponens deductions) as an intermediate step. Since we are not allowed to use these methodologies under Semantic Tableaux deductive calculi, our strategy for proving Theorem 1 will instead use Theorem 4's formalism as an alternate interim step to facilitate the proof.

We will now summarize the notation used in Theorem 1's proof. For any fixed integer i , let $G_i(x)$ denote the scalar-multiplication operation that

maps the integer x onto the quantity $2^{2^i} \cdot x$. Let $\Lambda(i, x, y)$ denote a Σ_0^* formula which indicates that $y = G_i(x)$. There are many possible Σ_0^* encodings for $\Lambda(i, x, y)$'s graph. Equation (5) provides one example:

$$\{i \neq 0 \wedge x \neq 0 \rightarrow \exists v \leq y [\text{LogLog}(v) = i \wedge \text{LogLog}(v - 1) < i \wedge \frac{y}{v} = x \wedge \frac{y - 1}{v} < x]\}$$

$$\text{AND} \quad \{ [i = 0 \vee x = 0] \rightarrow y = x + x \} \tag{5}$$

Let Υ_i denote Equation (6)'s Π_2^* sentence, which states that the operation that maps x onto the integer $G_i(x)$ is a total function.

$$\Upsilon_i \quad =_{\text{def}} \quad [\forall x \exists y \quad \Lambda(\bar{i}, x, y)] \tag{6}$$

G_i 's definition clearly implies $\forall i \forall x G_{i+1}(x) = G_i(G_i(x))$. Equation (7) encodes this fact *as a Π_1^* sentence*.

$$\forall i \forall x \forall y \forall z [\Lambda(i, x, y) \wedge \Lambda(i, y, z)] \rightarrow \Lambda(i + 1, x, z) \tag{7}$$

Also, for any $m > 0$ and $n > 0$, let (8) and (9) define the sentences Θ_m and \mathcal{U}_n below. (Equation (7) implies their validity.)

$$\Theta_m \quad =_{\text{def}} \quad [\Upsilon_{m-1} \rightarrow \Upsilon_m] \tag{8}$$

$$\mathcal{U}_n \quad =_{\text{def}} \quad [\Upsilon_0 \wedge \Theta_1 \wedge \Theta_2 \wedge \dots \wedge \Theta_n] \tag{9}$$

The intuitive reason why the Second Incompleteness Theorem is harder to prove for the Semantic Tableaux deductive calculus than for the Hilbert method of deduction can be appreciated by examining an axiom system β whose only *non-trivial* axiom about the U-Grounding functions corresponds to (7)'s axiom. This axiom, combined with the built-in assumption that Addition is a total function, allows one to construct a Hilbert-style proof of the theorem Υ_n from β whose length is no longer than $c \cdot n^c$ for some constant c , using a methodology that many logicians [4, 7, 8, 10, 11, 13, 16, 17, 21, 29, 32] have attributed to unpublished private communications from Robert Solovay. However, it oddly turns out that while the Semantic Tableaux calculus supports equally short proofs of similar approximate length $O(c \cdot n^c)$ for the sentences Θ_n and \mathcal{U}_n from β , there is no analogously short *Semantic Tableaux* proof of Υ_n from β !

Our proof of Theorem 1 is related to this fact. It will formalize a Level(2+) Tableaux-style generalization of the Second Incompleteness Theorem that has no analog at Level(1) essentially because \mathcal{U}_n has a much shorter proof than Υ_n from β (under Semantic Tableaux). No analog of this Tableaux-type separation characterizes the several Hilbert-style versions of the Second Incompleteness Theorem, discussed in say [4, 7, 8, 9, 10, 13, 14, 16, 17, 25, 26, 27, 29], where the Second Incompleteness Theorem is equally valid at all levels L . The intuitive reason the Semantic Tableaux will be shown in the discussion (below) to have contrasting properties for the Levels 1 and 2+ , unlike Hilbert deduction, will ultimately be because the difference between the proof-lengths of Υ_n and \mathcal{U}_n is much greater under Semantic Tableaux deduction than under Hilbert deduction.

In essence, the above observations combined with Theorem 4's role as a very helpful intermediate step explain the two main underlying intuitions behind Theorem 1's proof.

Summary of Main Proof: In the context of our proof of Theorem 1, the symbol “ α ” in the predicates $\text{ShortPrf}_\alpha^2(x, y, z)$ and $\text{Prf}_\alpha(x, y)$ will have a slightly unconventional interpretation. Rather than treat “ α ” as a fixed constant that denotes a finite-sized axiom system, this section will view it as an integer that designates a Gödel number that represents a finite sequence of sentences $S_1, S_2 \dots S_n$, listing α 's axioms. (Under our method for encoding an axiom system α , its Gödel number will have a bit-length proportional to the sum of the lengths of $S_1, S_2 \dots S_n$.)

Our formal analysis will begin by defining the Π_1^* sentence W mentioned in Theorem 1's hypothesis. It is defined as a conjunction of nine Π_1^* clauses $W_0, W_1 \dots W_8$ — where many of these clauses W_i are in turn conjunctions of several further Π_1^* sub-clauses. These nine clauses are defined below:

Definition of W_0 : This axiom will be a multi-clause Π_1^* sentence which provides sufficient information about the eight U-Grounding functions so that W_0 has the capacity to formally prove every Σ_0^* sentence that is logically valid. (It is unimportant which particular finite set of Π_1^* clauses is used to formulate W_0 , as long as one of its clauses is the explicit statement “ $0 \neq 1$ ”.)

Definitions of W_1 through W_5 : The definition of the Π_1^* sentence W_1 was given in Equation (7). In a context where $\text{Prf}_\alpha(x, y)$, $\text{Subst}(g, h)$ and $\Lambda(i, x, y)$ were already given Σ_0^* defining formulae, the further Π_1^* definitions

of W_2 through W_5 are given by Equations 10 through 13.

$$\forall g \forall h \forall h^* \{ [\text{Subst}(g, h) \wedge \text{Subst}(g, h^*)] \rightarrow h = h^* \} \quad (10)$$

$$\forall i \forall z [\Lambda(i, \bar{1}, z) \rightarrow \text{LogLog}(z) = i] \quad (11)$$

$$\forall \alpha \forall t \forall n [\text{Prf}_\alpha(t, n) \vee \neg \text{Prf}_\alpha(t, n)] \quad (12)$$

$$\forall g \forall h [\text{Subst}(g, h) \vee \neg \text{Subst}(g, h)] \quad (13)$$

Definitions of W_6 and W_7 : We will not provide a formal equational description of these two Π_1^* sentences, similar to the prior Equations (10) through (13), because their formal structures are a bit tedious to encode. Instead, we will provide a functional description of them:

1. The Π_1^* sentence W_6 will contain sufficient information about the U-Grounding functions so that for each ordered triple $(\bar{\alpha}, \bar{h}, \bar{y})$ where \bar{h} denotes the Gödel number of $D^2(\alpha)$ and $\text{Prf}_\alpha(h, y)$ is true, the formal proof of $\text{Prf}_\alpha(\bar{h}, \bar{y})$ from W_6 will have a bit-length no larger than $[\text{Log}(y)]^{C_6}$ for some constant C_6 . Likewise, the Π_1^* sentence W_6 will contain sufficient information about the U-Grounding functions so that for each ordered pair $(\bar{g}, \bar{h},)$ where $\text{Subst}(\bar{g}, \bar{h})$ is true, the formal proof of this fact will have a bit-length no larger than $[\text{Log}(h)]^{C_6}$ (It is easy to construct a Π_1^* axiom W_6 and accompanying constant C_6 with these properties.)
2. Let us recall that “U-Grounded Binary-encoded Representations” \bar{n} of integers n were defined in Section 2. The Π_1^* sentence W_7 will contain sufficient information about the U-Grounding functions so that for any integer $n > 1$, the proof from W_7 of “ $\overline{n-1} + \bar{1} = \bar{n}$ ” has a bit-length no larger than n^{C_7} , for some sufficiently large fixed constant C_7 .

Definition of W_8 : The Π_1^* definition of W_8 will appear in Equation (26) later in this section. Its presentation is postponed because some preliminary lemmas need to first help motivate it.

Lemma 1. *Each of W_0 through W_7 are Π_1^* theorems of Peano Arithmetic.*

Proof: It is obvious that W_1 through W_5 are Π_1^* theorems of Peano Arithmetic. Also, it is trivial to construct Π_1^* sentences W_0 , W_6 and W_7 that satisfy their functional requirements. \square

Lemma 2. *Let us recall that Υ_i , Θ_m and \mathcal{U}_n were defined by Equations (6), (8) and (9). Then there exists three constants K_0 , K_1 and K_2 such that:*

- i) A semantic tableaux proof of Θ_m from W requires a bit-length no greater than $K_1 \cdot m^{K_1}$.*
- ii) A semantic tableaux proof of \mathcal{U}_n from W requires a bit-length no greater than $K_2 \cdot n^{K_2}$.*
- iii) The semantic tableaux proofs of Υ_i from the union of W and Υ_{i-1} , and of $\neg \Upsilon_{i-1}$ from the union of W and $\neg \Upsilon_i$, each require a bit-length no greater than $K_0 \cdot i^{K_0}$.*

Proof:

It is immediate from the definition of the sentences W_1 and W_7 that these two parts of W are sufficient to assure that Θ_m 's proof will have a length satisfying constraint (i). The assertion (ii) follows from (i) because the proof of \mathcal{U}_n has a length essentially no greater than the sum of the proof lengths for Υ_0 and for $\Theta_1, \Theta_2 \dots \Theta_n$. The assertion (iii) follows from (i) because the definition of Θ_m makes it obvious that (iii)'s two proofs have a sufficiently similar structure to (i)'s proof for there to be no meaningful difference between their proof lengths.

\square

Our proof of Theorem 1 will be centered around showing that the axiom system α satisfies the requirement (B) of Theorem 4 when $\lambda = 2$. After establishing this fact, the remainder of Theorem 1's proof will be quite easy. Paraphrased into the English Language, Theorem 4's Part-B requirement, with $\lambda = 2$ and L representing a fixed constant, is the statement:

+ If an ordered pair (y, z) (with $z > L$) encodes a "2-short proof" (from α) of the Gödel-like diagonalization sentence $D^2(\alpha)$, then there will exist some corresponding Q_2^* sentence (called say S) where both S and $\neg S$ have Semantic Tableaux proofs from α whose Gödel numbers are smaller than z .

Our proof of this statement will rest on showing that for some integer n (whose exact value will depend only on the ordered pair (y, z)), one adequate sentence S satisfying the above assertion is Equation (9)'s sentence \mathcal{U}_n . In order to complete Theorem 1's proof, we will need to show that any axiom system satisfying Theorem 1's hypothesis *will both satisfy* $+$'s requirements *and recognize this fact about itself*.

Some added notation will be employed by our next two lemmas. The symbol W^* will denote a multi-clause axiom, similar to W , except that W^* 's clauses will consist of W_0 through W_7 (and thus omit the W_8 condition). Also, p will be called a **Partial Proof** of the theorem Φ from α iff its structure is identical to a Semantic Tableaux proof tree **except** that one of its branches is released from the requirement of containing a pair of contradictory nodes. This unique branch will be called p 's **Open Branch**. Its lowest node will be called p 's **Bottom Node**. Our proof of $+$ will have a nicely compartmentalized modular nature, where it develops a sequence of increasing complex partial proof trees P_1, P_2, \dots, P_m , where each tree P_{i+1} is an extension of the prior tree P_i and the final object P_m is the well-defined Semantic Tableaux proof of the sentence $\neg S$ required by Statement $+$.

Lemma 3. *There exists a constant $K_3 > 0$ (whose exact numeric value will be unimportant to our main theorem) such that for each $n \geq 1$, it is possible to construct a Partial Proof P_1 of $\neg \mathcal{U}_n$ from the axiom system W^* whose bottom node stores Υ_n and where P_1 's length is bounded by $K_3 \cdot n^{K_3}$.*

Proof: Since P_1 represents a proof of $\neg \mathcal{U}_n$, its root will consist of the sentence $\neg \neg \mathcal{U}_n$. The root's child will be the sentence \mathcal{U}_n (which is formally derived via Section 2's \neg Elimination rule for Semantic Tableaux proofs). Then via several applications of the \wedge Elimination rule, Equation (9)'s \mathcal{U}_n sentence will be broken repeatedly into smaller and smaller components until each of the formal Q_2^* sentences of $\Upsilon_0, \Theta_1, \Theta_2, \dots, \Theta_n$ is enumerated along P_1 's open branch. The last n steps of P_1 's proof will consist in chronological order of n repeated applications of the \rightarrow Elimination Rule, whose i -th iteration splits Equation (8)'s Θ_i sentence into a left sibling nodes of the form $\neg \Upsilon_{i-1}$ and a right sibling of Υ_i . (These splits will be performed so that the sentences $\Upsilon_1, \Upsilon_2, \dots, \Upsilon_n$ are enumerated in chronological order along P_1 's open branch.)

To verify the above tree is a "Partial Proof" whose "Bottom Node" is Υ_n , we need to confirm each of the preceding paragraph's "left sibling nodes" of

the form $\neg \Upsilon_{i-1}$ are indeed leaves lying at the bottom of closed branches. This fact is a consequence of our inductive construction because the parent of each leaf storing a sentence $\neg \Upsilon_{i-1}$ will store its negation Υ_{i-1} .

Hence, P_1 certainly represents a Partial Proof of $\neg \mathcal{U}_n$. It is trivial that for some constant $K_3 > 0$, its proof length is bounded by $K_3 \cdot n^{K_3}$. \square

Lemma 4. *There exists a constant $K_4 > 0$ such that for every $n \geq 1$ it is possible to construct a Partial Proof P_2 of the sentence $\neg \mathcal{U}_n$ from the axiom system W^* — where this proof's bit-length is bounded by $K_4 \cdot n^{K_4}$ and where for some parameter u (created during existential quantifier elimination) the Bottom Node of this Partial Proof is the sentence “ $\text{LogLog}(u) = \bar{n}$ ”.*

Proof: The proof P_2 is constructed by taking Lemma 3's partial proof P_1 (whose Bottom node had stored the sentence Υ_n) and adding seven nodes below this bottom node. The first two of these seven nodes will store the sentences indicated by Equations (14) and (15). In particular, (14) is deduced from Equation (6)'s sentence Υ_n via the \forall Elimination Rule, and (15) is deduced from (14) via the \exists Elimination Rule.

$$\exists y \quad \Lambda(\bar{n}, \bar{1}, y) \tag{14}$$

$$\Lambda(\bar{n}, \bar{1}, u) \tag{15}$$

The next three sentences in P_2 's proof tree will consist of the axiom W_3 (whose formal statement was listed in Equation (11)) and then two deductions from this axiom based on applying the \forall Elimination Rule so that Equation (11)'s universally quantified variables of i and z are replaced by \bar{n} and u . The final sentence at the bottom of these three steps is:

$$\Lambda(\bar{n}, \bar{1}, u) \rightarrow \text{LogLog}(u) = \bar{n} \tag{16}$$

The last two nodes of P_2 's proof tree will be deduced from (16) via the \rightarrow Elimination rule. These two nodes will thus be sibling nodes storing the sentences of “ $\neg \Lambda(\bar{n}, \bar{1}, u)$ ” and “ $\text{LogLog}(u) = \bar{n}$ ”.

The tree P_2 (above) is a Partial Proof because “ $\text{LogLog}(u) = \bar{n}$ ” is its Bottom Node and the vertex storing “ $\neg \Lambda(\bar{n}, \bar{1}, u)$ ” is contradicted by Equation (15)'s sentence (and hence represents the needed leaf closing a branch). Since P_2 differs from P_1 by only having seven additional nodes,

both trees have the same approximate bit-length (thereby establishing that P_2 's length is sufficiently small to satisfy Lemma 4's claim). \square

Lemma 5. *Let α denote any axiom system of finite cardinality that is an extension of W^* . Let us recall the Gödel Sentence $D^2(\alpha)$ was defined in Section 3. Suppose N denotes the Gödel number of a semantic tableaux proof P of the sentence $\neg \mathcal{U}_N$ from α whose bit-length is bounded by $K_5 \cdot N^{K_5}$ for some fixed constant $K_5 > 0$.*

Proof Sketch. The proof P is built by taking Lemma 4's partial proof P_2 and inserting directly below P_2 's Bottom Sentence nine additional nodes and three further subtrees. The first of these nine nodes will be Equation (12)'s W_4 axiom sentence. The next three nodes will represent reductions from this axiom, using the \forall Elimination Rule to replace (12)'s universally quantified variables t , α and n with $D^2(\alpha)$'s Gödel number, $\bar{\alpha}$ and \bar{N} . The resulting sentence at the end of these reductions is:

$$\text{Prf}_{\bar{\alpha}} ([D^2(\alpha)] , \bar{N}) \vee \neg \text{Prf}_{\bar{\alpha}} ([D^2(\alpha)] , \bar{N}) \quad (17)$$

Below node (17), P 's proof will apply the \vee Elimination rule to produce the following two sibling nodes

$$\neg \text{Prf}_{\bar{\alpha}} ([D^2(\alpha)] , \bar{N}) \quad (18)$$

$$\text{Prf}_{\bar{\alpha}} ([D^2(\alpha)] , \bar{N}) \quad (19)$$

It is easy to inject a closed subtree T_1 below node (18) which meets Lemma 5's requirements. This is because the lemma's hypothesis indicates N is a proof of $D^2(\alpha)$ and the axiom W_6 then implies the subtree T_1 can have a small enough bit-length to satisfy Lemma 5's requirements.

In order to construct an analogous second suitably small subtree below (19)'s sentence, let G denote the particular Gödel number satisfying the identity $\text{Subst} (G , [D^2(\alpha)])$. In this context, we will insert below (19)'s sentence, a 3-node sequence beginning with Equation (13)'s W_5 axiom followed by two iterations of the \forall Elimination rule to obtain the deduction:

$$\text{Subst} (\bar{G} , [D^2(\alpha)]) \vee \neg \text{Subst} (\bar{G} , [D^2(\alpha)] ,) \quad (20)$$

Next, let us apply the \vee Elimination Rule to split (20)'s node into (21) and (22)'s pair of sibling nodes.

$$\neg\text{Subst} (\overline{G} , [D^2(\alpha)] ,) \quad (21)$$

$$\text{Subst} (\overline{G} , [D^2(\alpha)] ,) \quad (22)$$

Using axiom W_6 , it is easy to insert below node (21) a closed subtree T_{2A} whose bit-length is sufficiently small to satisfy Lemma 5's requirements. Thus to complete Lemma 5's proof, we must merely show that it is also possible to insert below node (22) a second adequately small subtree T_{2B} .

In order to construct T_{2B} , we will use again the fact that N represents a proof of the theorem $D^2(\alpha)$. Since N proves a theorem $D^2(\alpha)$ (whose formal statement in Equation (3) begins with three universally quantified variables y, z and h), one can apply standard methods from Proof Theory to deduce the existence of a proof T^* of $0=1$ from the union of the axiom system α with the added axioms given in Equations (23) through (25) such that $T^* \leq N^2$ when these two proofs are viewed as Gödel numbers. (The footnote ¹ explains the intuition behind T^* 's construction, and a longer version of this paper will give a more formal proof that $T^* \leq N^2$ exists.)

$$\text{Subst} (\overline{G} , u_1) \quad (23)$$

$$\text{Prf}_{\overline{\alpha}} (u_1 , u_2) \quad (24)$$

$$\text{LogLog}(u) = u_2 \quad (25)$$

In the above context, T_{2B} will be defined as a tree identical to T^* except that each appearance of u_1 and u_2 in T^* is replaced by respectively $[D^2(\overline{\alpha})]$

¹The intuitive reason that $T^* \leq N^2$ exists is that a proof N of $D^2(\alpha)$ will store $\neg D^2(\alpha)$ in N 's root, and the main non-degenerate versions of such proofs N will next apply the \neg Elimination Rule to transform $D^2(\alpha)$'s universally quantified variables y, z and h into existentially quantified variables that are subsequently replaced by the parameter symbols u_1, u_2 and u satisfying Equations (23) through (25) in a context where these three nodes appear in a straight-line path in N 's proof-tree directly below the root. This footnote should not be considered a formal proof that there exists the required $T^* \leq N^2$ because a formal proof, appearing in a longer version of this paper, must also consider certain degenerate cases, in addition to the main non-degenerate case outlined here.

and \overline{N} in T_{2B} . This subtree must certainly close the portion of P 's proof tree that descends from node (22) because the sentences in Equations (23) through (25) are identical to the three respective sentences in (22), (19) and P_2 's Bottom Node with $[D^2(\overline{\alpha})]$ and \overline{N} now replacing u_1 and u_2 . Essentially because the replacement of u_2 with the longer expression of \overline{N} will cause T_{2B} 's bit-length to have an $O[(\text{Log } N)^2]$ magnitude, the resulting constructed T_{2B} tree will be short enough to satisfy Lemma 5's claim. \square

Lemma 6. *Let $\text{MinAx}(\alpha)$ denote a Σ_0^* formula that indicates α is an axiom system of finite cardinality which includes the Π_1^* sentences of $W_0, W_1 \dots W_7$. Then there exists a suitably large constant $L > 0$ such that Equation (26)'s Π_1^* sentence is both valid and a theorem of Peano Arithmetic:*

$$\forall z > L \forall y \forall r \{ [\text{ShortPrf}_r^2([D^2(r)], y, z) \wedge \text{MinAx}(r)] \rightarrow$$

$$\exists p < z \exists q < z \exists s < z \exists t < z [\text{Pair}(s, t) \wedge \text{Prf}_r(p, s) \wedge \text{Prf}_r(q, t)] \} \quad (26)$$

Proof: The combination of Lemma 2 (part ii) and Lemma 5 easily implies that Equation (26) is valid for sufficiently large enough L . This is because these two lemmas imply that if (z, y, r) satisfies the left side of (26)'s implication clause, then (for suitably large enough L) the Q_2^* sentence \mathcal{U}_y will have short enough proofs of both itself and its negation to automatically satisfy the right side of (26)'s implication clause. (Moreover, (26) is a Π_1^* theorem of Peano Arithmetic because all the lemmas presented in this chapter can be formally proven by Peano Arithmetic.) \square

Finishing the Definition of W . The final clause W_8 of the system W will be defined as Equation (26)'s Π_1^* sentence with a large enough value assigned to the constant L to make this sentence valid. (We had not provided W_8 's definition earlier in this section because it seemed more appropriate to first introduce the Lemma 6 — indicating the correctness of W_8 — before defining it.)

Finishing the Proof of Theorem 1. The combination of Lemmas 1 through 6 indicates that all the clause of W are valid Π_1^* theorems of Peano Arithmetic (as the hypothesis of Theorem 1 had required). To finish the proof of Theorem 1, we must show that every finite and consistent axiom system $\alpha \supset W$ is unable to prove its Level(2+) consistency.

This fact is an easy consequence of Theorem 4. In particular, any $\alpha \supset W$ must be able to prove the two sentences (A) and (B) required by Theorem 4 because the sentence (A) is identical to α 's W_2 axiom clause and because α can verify (B) by taking Equation (26)'s W_8 axiom-sentence and observing that it reduces to (B) when one sets $r = \bar{\alpha}$ (and uses the fact that $\text{MinAx}(\bar{\alpha}) = \text{True}$). Hence since α can prove (A) and (B), Theorem 4 indicates α is unable to verify its Level(2+) Tableaux consistency. \square

Clarification Concerning Theorem 1's Proof and Meaning. The preceding paragraph showed how it was relatively easy to derive from Theorem 4 the conclusion that no consistent $\alpha \supset W$ can verify its own Level(2+) Tableaux consistency. This result would actually *be totally meaningless* if W was an invalid statement because there would then be no example of a consistent $\alpha \supset W$ *actually existing*! The more subtle aspect of Theorem 1's proof was thus not its second paragraph (which explored the properties of systems satisfying $\alpha \supset W$) but rather it was its first paragraph (which noted Lemmas 1 through 6 imply W is a logically valid Π_1^* sentence). Without this latter crucial fact, Theorem 1 would be devoid of meaningful, non-trivial structural implications. (A similar distinction about the importance of W being a valid Π_1^* sentence will also apply to the generalizations of Theorem 1 explored in the next section.)

5 Sketch of Theorem 2's Proof

Throughout this section, $\text{Tab-}\mathfrak{R}\text{-List}_\alpha^\lambda(t, p)$ will denote a Σ_0^* sentence asserting p is a $\text{Tab-}\mathfrak{R}\text{-List}$ proof of the theorem t from the axiom system α . Also, $\text{Short-}\mathfrak{R}\text{-List}_\alpha^\lambda(x, y, z)$ will denote the TabList analog of Section 3's $\text{ShortPrf}(x, y, z)$ predicate. It will thus denote a Σ_0^* sentence that asserts that the two conditions of $y = \text{Log}^\lambda(z)$ and $\text{Tab-}\mathfrak{R}\text{-List}_\alpha(x, y)$ both hold. Also, $D_{\mathfrak{R}}^\lambda(\alpha)$ will denote the analog of Section 3's $D^\lambda(\alpha)$ Gödel sentence. Thus, $D_{\mathfrak{R}}^\lambda(\alpha)$ is defined to be the following diagonalization sentence:

There exists no code (y, z) that represents a ShortList proof (with exponent λ and intermediate set \mathfrak{R}) of **this sentence** from α 's axioms.

Throughout this section, the symbol \perp will denote the Gödel number of the sentence $0 = 1$. Our general technique for proving Theorem 2 will

employ a methodology very similar to Theorem 1's proof. It will thus employ an intermediate step, analogous to the Theorem 4, used in Sections 3 and 4. The formal statement of this revised form of Theorem 4 is given below:

Theorem 5. . *Let \mathfrak{R} denote an arbitrary class of sentences. (In our particular applications of Theorem 5, \mathfrak{R} will represent either the set of Π_2^* formulae or the set of Σ_2^* formulae). Suppose α is a consistent axiom system capable of proving all the Σ_0^* sentences that are valid in the Standard Model. Suppose there exists two constants, λ and L , such that α can also prove:*

$$\begin{aligned} A) \quad & \forall g \forall h \forall h^* \{ [\text{Subst}(g, h) \wedge \text{Subst}(g, h^*)] \rightarrow h = h^* \} \\ B) \quad & \forall z > L \quad \forall y \quad \{ \text{Short-}\mathfrak{R}\text{-List}_\alpha^\lambda([D_{\mathfrak{R}}^\lambda(\alpha)] , y, z) \rightarrow \\ & \exists p < z \quad \text{Tab-}\mathfrak{R}\text{-List}_\alpha(\perp , p) \} \end{aligned}$$

Then α must be incapable of proving Equation (27)'s theorem statement (which intuitively indicates that Tab- \mathfrak{R} -List proofs using α 's set of proper axioms are Level(0-) consistent).

$$\forall p \quad \neg \text{Tab-}\mathfrak{R}\text{-List}_\alpha(\perp , p) \tag{27}$$

In addition to using Theorem 5 to prove Theorem 2, we will need an approximate analog of the prior section's Lemma 6. Below is Lemma 6's analog for TabList deduction:

Lemma 7. *Let $\text{Min}W(\alpha)$ denote a Σ_0^* formula that indicates α is an axiom system of finite cardinality which includes Section 4's Π_1^* sentence W . Then there exists two suitably large constants $L_1 > 0$ and $L_2 > 0$ such that Equations (28) and (29) are both Π_1^* theorems of Peano Arithmetic:*

$$\begin{aligned} \forall z > L_1 \forall y \forall r \quad & \{ [\text{Short-}\Pi_2^*\text{-List}_r^2([D_{\Pi_2^*}^2(r)] , y, z) \wedge \text{Min}W(r)] \rightarrow \\ & \exists p_1 < z \quad \text{Tab-}\Pi_2^*\text{-List}_\alpha(\perp , p_1) \} \end{aligned} \tag{28}$$

$$\begin{aligned} \forall z > L_2 \forall y \forall r \quad & \{ [\text{Short-}\Sigma_2^*\text{-List}_r^2([D_{\Sigma_2^*}^2(r)] , y, z) \wedge \text{Min}W(r)] \rightarrow \\ & \exists p_2 < z \quad \text{Tab-}\Sigma_2^*\text{-List}_\alpha(\perp , p_1) \} \end{aligned} \tag{29}$$

Proof Sketch for Lemma 7. In a very abbreviated form, the proof element p_1 needed to make Equation (28) valid can be summarized as having the following 4-part structure:

1. Its first fragment will be the Tab- Π_2^* -List proof y , which intuitively represents a proof of the Π_1^* theorem $D_{\Pi_2^*}^2(\alpha)$.
2. Its second fragment will prove the Π_0^* theorem which states that \bar{y} , viewed as a Gödel number, proves the theorem $D_{\Pi_2^*}^2(\alpha)$. (This theorem is formally encoded as: Tab- Π_2^* -List $^2_\alpha(\lceil D_{\Pi_2^*}^2(r) \rceil, \bar{y})$.)
3. Its third fragment will prove the precise sequence of Π_2^* theorems of $\Upsilon_1, \Upsilon_2, \dots, \Upsilon_y$ in exactly the chronological order just specified. (The reason it is necessary to prove these theorems in increasing chronological order is that it will turn out that Part-iii of Lemma 2 can then assure that each proof is sufficiently compact for p_1 to satisfy Equation (28)'s severe size constraint of $p_1 < z$.)
4. The last fragment of p_1 will use the combination of the preceding three results to prove $0=1$, via a variant of Gödel's traditional diagonalization contradiction argument. (The exactly helpful role of the theorem Υ_y in this contradiction proof is that it will guarantee the existence of the integer $z = 2^{2^y}$ — whose formally proven existence is required to finish our proof of $0=1$.)

Likewise summarized again in an abbreviated form, the proof element p_2 satisfying Equation (29) is the following 4-part structure:

1. Its first fragment will be the Tab- Σ_2^* -List proof y , which intuitively represents a proof of the Π_1^* theorem $D_{\Sigma_2^*}^2(\alpha)$.
2. Its second fragment will prove the Σ_0^* theorem which states that \bar{y} , viewed as a Gödel number, proves the theorem $D_{\Sigma_2^*}^2(\alpha)$. (This Σ_0^* theorem is formally encoded as: Tab- Σ_2^* -List $^2_\alpha(\lceil D_{\Sigma_2^*}^2(r) \rceil, \bar{y})$.)
3. Its third fragment will prove the precise sequence of Σ_2^* theorems of $\neg \Upsilon_y, \neg \Upsilon_{y-1}, \dots, \neg \Upsilon_1$ in exactly the descending chronological order just specified. (The reason it is necessary to prove these theorems in decreasing chronological order is that the theorem $\neg \Upsilon_y$ can be derived

easily as a combined consequence of Item 1 and 2's intermediate results, and for each integer i we can then use Part-iii of Lemma 2 to obtain a suitably short proof of $\neg \Upsilon_{i-1}$ from the preceding intermediate result of $\neg \Upsilon_i$ to satisfy Equation (29)'s severe size constraint of $p_2 < z$.)

4. The last fragment of p_2 will use Item 3's intermediate result of $\neg \Upsilon_1$ to prove the desired contradiction theorem of $0=1$.

The proof that the objects p_1 and p_2 satisfy the size constraints in Lemma 7's two claims can be summarized as being analogous to Section 4's multi-step proof of Lemma 6 — except that roughly Part-iii of Lemma 2 will now replace its prior Part-ii in certain intermediate steps of Lemma 7's proof. (A more formal proof of Lemma 7 will appear in a longer version of this article.)

Sketch of the Remainder of Theorem 2's proof: To complete Theorem 5's proof, we will set V_A equal to the conjunction of Section 4's axiom W with Equation (28)'s Π_1^* sentence, and set V_B equal to the conjunction of W with Equation (29)'s Π_1^* sentence. Then a similar type of diagonalization proof, as was used in Section 4, can finish Theorem 2's proof. In particular, one can combine the intermediate results of Theorem 5 and Lemma 7 to derive Theorem 2 in the same manner that the prior section combined Theorem 4 and Lemma 6 to derive Theorem 1. (Several additional details concerning exactly how Section 4's proof of Theorem 1 can generalize to also verify Theorem 2 shall appear in a longer version of this paper.)

Significance of Theorem 2's Result. Part of the reason that Theorem 2 is significant is that Theorem 3 shows that if one were to weaken only slightly either Part-A or Part-B of Theorem 2's hypothesis then significant Boundary-Case Exceptions to the Second Incompleteness Theorem will arise. It should also be stated that Theorem 2 generalizes to all axioms systems of infinite cardinality satisfying Remark 1's Conventional Deciphering Property.

It is also apparent that the Incompleteness effects described by Theorems 1 and 2 can generalize from Semantic Tableaux deduction to any other cut-free rule of inference, such as for example Herbrand deduction or the cut-free variants of the Sequent calculus. (In the particular case of Theorem 2, the analogs of TabList deduction obviously will have their subcomponent proofs p_1, p_2, \dots, p_n consist of Herbrand or Cut-Free sequent calculus proofs.)

6 Appendix: Sketch of Theorem 3's Proof

This appendix will sketch a proof of Theorem 3. As Section 2 had explained, Theorem 3's result was technically announced on the last page of our Tableaux-2002 paper [34]. However, the latter conference paper only formally proved the correctness of a weaker version of this result — where Semantic Tableaux deduction replaced Tab₁List deduction in Clause 1 of Theorem 3's formal statement. Our goal in this Appendix is to briefly summarize the added functionality needed to prove the stronger and more general version of [34]'s theorem. Throughout this appendix, we will assume that the reader has already examined our prior paper [34] and has a copy of it on his desk. Thus, we will not review most of the definitions from [34]. Also, if a Lemma number ends with “t” (as in say “Lemma 2-t”) then it refers to a result from our Tableaux–2002 Conference paper [34].

In all candor, we first hesitated to include a very abbreviated proof of Theorem 3 as an appendix to this article. However, it seemed desirable to include some type of justification of Theorem 3 here because there is such a tight match between Theorem 3's positive result and Theorem 2's complementing negative result for this topic to be worth mentioning.

Notation Conventions: The deductive rule of inference that had been called “R(1,1) Hierarchy Deduction” in our prior paper [34] has now been renamed, and it is called instead “Tab₁List” deduction in the current paper. Thus Tab₁List_α(*t*, *p*) will denote a Σ₀^{*} formula indicating that *p* is a Tab₁List proof of the theorem *t* from the axiom system α

As in our earlier article [34], Pair^{*}(*x*, *y*) will denote a Σ₀^{*} formula indicating that *x* is a Π₁^{*} sentence and *y* is its negation. The last page of [34] had defined IS-1^{*}(*A*) as an axiom system identical to [34]'s IS-1(*A*) system, except that Tab₁List deduction had replaced Semantic Tableaux deduction in the statement of IS-1^{*}(*A*)'s self-justifying Group-3 axiom. Thus, the Group-3 axiom of IS-1^{*}(*A*) is a self-referencing axiom of the form:

$$\forall x \forall y \forall p \forall q \quad \neg [\text{Pair}^*(x, y) \wedge \text{Tab}_1\text{List}_{\text{IS-1}^*(A)}(x, p) \wedge \text{Tab}_1\text{List}_{\text{IS-1}^*(A)}(y, q)] \quad (30)$$

Our proof of Theorem 3 will rest on showing that one can generalize the Theorem 2-t from [34] to establish that:

+ If *A* is consistent then IS-1^{*}(*A*) is automatically also consistent.

Henceforth, $\omega(x, y, p, q)$ will denote the Σ_0^* formula enclosed within (30)'s square bracket expression. In order to establish $++$, it is sufficient to prove the following alternate form of this assertion:

$++$ If A is consistent then $\forall x \forall y \forall p \forall q \neg \omega(x, y, p, q)$

Outline of the Proof of $++$. For the sake of constructing a proof-by-contradiction, let us assume $++$ was false and that (X, Y, P, Q) denotes the particular tuple satisfying $\omega(X, Y, P, Q)$ that has minimal value for the quantity $G = \text{Max}(P, Q)$. Then Equation (31) is valid:

$$\forall x \forall y \forall p \forall q \{ p < G \wedge q < G \} \rightarrow \neg \omega(p, q, x, y) \quad (31)$$

The procedure PROBE, the notion of a (L, M) -Conservative Branch of a Semantic Tableaux proof and the condition called $\text{Constraint}(p, \beta)$ were each defined in [34]. Our goals will be 1) to use Equation (31) and these concepts to construct an ordered pair satisfying $\text{Constraint}(t, \beta)$, and then 2) to combine this fact with [34]'s Lemma 1-t to finish our proof-by-contradiction.

Some added notation is now needed to formalize this proof of $++$. Let H again denote a Tab_1List proof, comprised of the sequence of ordered pairs of $(t_1, p_1), (t_2, p_2) \dots (t_n, p_n)$, where p_i is a semantic Tableaux proof of the intermediate result t_i . Let us call Υ the **Conclusion** of the proof H iff it is the last theorem that H proves (i.e. this means that $t_n = \Upsilon$). Define $\chi(p_i)$ to be the number of logical symbols appearing in p_i 's proof. Also, let $\mathfrak{S}(H)$ denote the quantity $\sum_{i=1}^n \chi(p_i)$. Let us say that:

1. The Σ_1^* sentence “ $\exists v_1 \exists v_2 \dots \exists v_m \psi(v_1, v_2, \dots v_m)$ ” is G -good iff there exists a Tab_1List proof $H \leq G$ from $\text{IS-1}^*(A)$ of this sentence, and it is accompanied with a valid Equation (32).

$$\exists v_1 \leq 2^{\mathfrak{S}(H)} \exists v_2 \leq 2^{\mathfrak{S}(H)} \dots \exists v_m \leq 2^{\mathfrak{S}(H)} \psi(v_1, v_2, \dots v_m) \quad (32)$$

2. The Π_1^* sentence “ $\forall v_1 \forall v_2 \dots \forall v_m \psi(v_1, v_2, \dots v_m)$ ” is G -good iff there exists a Tab_1List proof $H \leq G$ from $\text{IS-1}^*(A)$ of this sentence, and it is accompanied with a valid Equation (33).

$$\forall v_1 \leq G \cdot 2^{-\mathfrak{S}(H)} \forall v_2 \leq G \cdot 2^{-\mathfrak{S}(H)} \dots \forall v_m \leq G \cdot 2^{-\mathfrak{S}(H)} \phi(v_1, v_2, \dots v_m) \quad (33)$$

The opening paragraph of this proof had assumed that $\omega(X, Y, P, Q)$ was true, and it had defined the integer G to equal $\text{Max}(P, Q)$. In this context, all the standard encoding conventions imply that the two formal inequalities of $\mathfrak{S}(P) < \frac{1}{3} \text{Log}_2(G)$ and $\mathfrak{S}(Q) < \frac{1}{3} \text{Log}_2(G)$ both hold when the Π_1^* sentence X and the Σ_1^* sentence Y are both G -good. Since it is impossible for both X and Y to have G -good proofs under the preceding circumstances, we are forced to conclude that there exists some sentence Υ^* which fails to be G -good and whose proof $H^* \leq G$ has the smallest Gödel number among the set of proofs that fail the G -good criteria.

Let p^* denote the particular tableaux proof belonging to H^* that proves the theorem Υ^* . It turns out that [34]'s procedure PROBE can construct under these assumptions a branch β^* of p^* satisfying $\text{Constraint}(p^*, \beta^*)$. In particular, this procedure PROBE will produce such an output when it is given the input arguments of

1. $M = G \cdot 2^{-\mathfrak{S}(H^*)} - 1$, $L = 2^{\mathfrak{S}(H^*) - \chi(p^*)}$ and $T = p^*$ when Υ^* is a Π_1^* sentence.
2. $M = G \cdot 2^{\chi(p^*) - \mathfrak{S}(H^*)} - 1$, $L = 2^{\mathfrak{S}(H^*)}$ and $T = p^*$ when Υ^* is a Σ_1^* sentence.

The formal proof that these input values for L , M and T will enable the procedure PROBE to produce an (L, M) -Conservative branch is roughly similar to the 8-step construction used in [34] to justify its Lemmas 2-t and 3-t. The added details needed to now justify our stronger effect appear in a longer version of this paper. It differs from the analogous results from [34] by essentially using Equation (31) and the certifiable fact that H^* is the smallest Tab_1List style proof failing G -goodness to enable us to strengthen Lemmas 2-t and 3-t.

Our justification of $++$ can now be finished with the same type of proof-by-contradiction that was used in [34]. The preceding four paragraphs have shown that if $++$ was false, then we could construct some β satisfying $\text{Constraint}(p^*, \beta^*)$. However such a construction is actually impossible because the axioms in p^* 's proof would then form a system satisfying the hypothesis of Lemma 1-t. In this context, Lemma 1-t *certainly forbids* $\text{Constraint}(p^*, \beta^*)$ from being true. Hence, $++$ must be true to avoid this inherent contradiction. \square

Acknowledgment: I thank Robert Solovay for several stimulating telephone conversations that we had during 1994. In essence, the Theorems 1 and 2 of the current article were designed to strengthen the formalism from the Theorem * that Solovay communicated to me during 1994 (see [21]).

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- of [32] for a much longer 4-page summary of these telephone conversations with Robert Solovay and an interpretation of how the joint work of Nelson, Pudlák, Solovay and Wilkie-Paris had collectively produced the basic result of Theorem * .
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