Some Specially Formulated Axiomizations for $\Sigma_0$ Manage to Evade the Herbrandized Version of the Second Incompleteness Theorem

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Abstract

In 1981, Paris and Wilkie [28] indicated it was an open question whether $\Sigma_0$ would satisfy the Second Incompleteness Theorem for Herbrand deduction. We will show that some specially formulated axiomizations for $\Sigma_0$ can evade the Herbrandized version of the Second Incompleteness Theorem.

Key words: Gödel's Second Incompleteness Theorem, Herbrand Consistency
MSC: 03B52; 03F25; 03F45; 03H13

1 Introduction

Gödel's Second Incompleteness Theorem [11] asserts that neither Peano Arithmetic, nor any consistent extension of it, can prove a theorem affirming its own self-consistency under Hilbert deduction. There have been numerous generalizations and extensions of Gödel's seminal result [1–4,6–9,15,19,23,24,28,30–36,38,41,40,43,44,46,48,50,52,54]. For example, Solovay [36] has combined the work of Nelson, Pudlák and Wilkie-Paris [25,31,46] to establish that essentially no axiom system that recognizes $\text{Successor}(x) = x + 1$ as a total function (and which treats addition and multiplication as 3-way relations) can prove a theorem affirming its own consistency under the conventional paradigm of Hilbert deduction.

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Preprint submitted to Elsevier 8 February 2013
In 1981, Paris and Wilkie [28] noticed that it was an open question whether
the axiom system $\mathbf{I} \Sigma_0$ did satisfy the Second Incompleteness Theorem for
semantic tableaux and Herbrand deduction. Interestingly at the same time,
Paris-Wilkie observed that there was an available solution to this problem for
Hilbert deduction. Thus, $\mathbf{I} \Sigma_0 + \text{Exp}$ is unable to prove the Hilbert consistency of
even an axiom system as simple as $\mathbf{Q}$ [46]. Subsequently, Adamowicz-Zbierski
[1,3] showed that $\mathbf{I} \Sigma_0 + \Omega_1$ was unable to verify its Herbrand consistency, and
Willard [48,50] developed an alternate variant of the Adamowicz-Zbierski for-
malism that showed at least one type of natural encoding of the $\mathbf{I} \Sigma_0$ axiom
system would satisfy the semantic tableaux version of the Second Incomple-
ness Theorem.

On 16 November 2005, we received a fascinating email communication from
L.A. Kołodziejczyk about this subject (which were an outgrowth out of some
conversations he had with Zofia Adamowicz and Konrad Zdanowksi). It ob-
served that there are two natural formalisms for axiomatizing $\mathbf{I} \Sigma_0$, henceforth
called Ax-1 and Ax-2. Both shall take the Tarski-Mostowski-Robinson axiom
system $\mathbf{Q}$ as their starting base. In a context where $\phi(x, y)$ is a $\Delta_0$
formula, these formalisms will use respectively Equations (1) and (2) to denote their
induction schemes.

\[
\forall x \left\{ \{ \phi(x, 0) \land \forall y [ \phi(x, y) \rightarrow \phi(x, y') ] \} \implies \forall y \phi(x, y) \right\} \quad (1)
\]

\[
\forall x \forall z \left\{ \{ \phi(x, 0) \land \forall y \leq z [ \phi(x, y) \rightarrow \phi(x, y') ] \} \implies \forall y \leq z \phi(x, y) \right\} \quad (2)
\]

Kołodziejczyk noticed that logically equivalent axiom systems, such as Ax-1
and Ax-2, do not necessarily have the same properties with regards to the
semantic tableaux and Herbrandized versions of the Second Incompleteness
Theorem. Thus, Kołodziejczyk’s email asked whether [50]’s semantic tableaux
version of the Second Incompleteness Theorem will generalize for Ax-2’s un-
conventional induction scheme? It also inquired to what extent are generaliza-
tions of the Second Incompleteness Theorem germane to Herbrand deduction?
One reason the second question is especially intriguing is that Kołodziejczyk
demonstrated in [18,19] that there exists bounded arithmetics where Herbrand
consistency and tableaux consistency are provably not logically equivalent.

One part of our answer to this question had appeared in the separate pa-
per [55]. It explained how our prior results about Ax-1’s semantic tableaux
incompleteness properties have direct generalizations for Ax-2.

A second type of answer to Kołodziejczyk’s stimulating open question will
appear in this paper. We will prove that there is a third type of axiomization
of $\mathbf{I} \Sigma_0$, called Ax-3, which is logically equivalent to Ax-1 and Ax-2, but which
has the property that an extension of Ax-3 is capable of recognizing its own
Herbrand consistency.
This last result is likely to raise almost as many questions as it does answer. This is because there are many potential logically equivalent axiomizations for $\text{I}\Sigma_0$. Thus, one may ask for which potential particular axiomizations $\alpha$ for $\text{I}\Sigma_0$ do the Questions (1) and (2) below have a positive answer?

(1) Are all extensions of $\alpha$’s axiomization for $\text{I}\Sigma_0$ unable to prove a theorem verifying their own semantic tableaux consistency?

(2) Likewise, are all extensions of $\alpha$’s axiomization for $\text{I}\Sigma_0$ unable to prove a theorem verifying their own Herbrand consistency?

Since the Second Incompleteness Theorem generalizes for most types of axiom systems, it is of course reasonable to conjecture that most of the answers to the Questions 1 and 2 (above) will be in a positive direction. However, the point of this article is that an automatic “yes” response to Questions 1 and 2 cannot be always secured. Thus, Kołodziejczyk [18,19] has shown that Herbrand consistency and semantic tableaux consistency are not always equivalent to each other, and the current article will actually construct a particular formalization of $\text{I}\Sigma_0$, called Ax-3, that manages to evade at least Question 2’s Herbrandized version of the Second Incompleteness Theorem.

2 The Definition of a New Version of $\text{I}\Sigma_0$

This section will define the Ax-3 axiomatization for $\text{I}\Sigma_0$ and provide the formal statement of our main theorem (which will be subsequently proven in Sections 3 and 4). As the reader examines the formal definitions in the current section, it should be kept in mind that Ax-3 was deliberately endowed with an unconventional method for axiomatizing $\text{I}\Sigma_0$ so that its formalism will evade the Herbrandized version of the Second Incompleteness Theorem. (Moreover because of a technical difference between the definitions of Herbrand consistency and semantic tableaux consistency, it should be kept in mind that Ax-3’s evasion of the Second Incompleteness Theorem under a Herbrandized definition of consistency does not generalize for semantic tableaux deduction.)

In our discussion, a formula will be called $\Delta^R_0$ iff it has a structure similar to a $\Delta_0$ formula (appearing in for example the Hájek -Pudlák textbook [13]) except that its bounded quantifiers, “$\forall v \leq T$ ” and “$\exists v \leq T$, ”, are now disallowed from using the conventional arithmetic functions of addition and multiplication in their terms $T$. Instead, the terms of a $\Delta^R_0$ formula will employ only the maximum function as the only permissible operator to define a variable’s bounded range. (Arithmetic functions are allowed to appear elsewhere in the body of a $\Delta^R_0$ formula.) Thus, Equation (3) is an example
of a $\Delta^R_0$ formula, and (4) is an example of a $\Delta_0$ formula that is not $\Delta^R_0$.

\[ \forall p \leq \text{Max}(x, y) \ [ ( p + y \leq x + 2 \cdot y ) \lor ( p \cdot y \leq y \cdot y \cdot y ) ] \] (3)

\[ \forall p \leq x \cdot y \ [ ( p + y \leq x + 2 \cdot y ) \lor ( p \cdot y \leq y \cdot y \cdot x ) ] \] (4)

Let us say a formula is $\Pi^R_1$ iff it can be written as $\forall v_1 \forall v_2 \ldots \forall v_n \phi(v_1, v_2, \ldots, v_n)$ where $\phi(v_1, v_2, \ldots, v_n)$ is a $\Delta^R_0$ formula. Each of Ax-1, Ax-2 and Ax-3 will contain a common set of nine $\Pi^R_1$ axioms, called $Q_0$ and listed below. The main purpose of $Q_0$ will be to define the constructs of addition, multiplication, integer-successor, maximum and also $=$ and $\leq$.

\[ 1 = 0' \land 2 = 1' \land 0 = 0 \land 0' \neq 0 \land 0 \leq 0 \land \neg [0' \leq 0] \] (5)

\[ \forall x \ ( \ x + 0 = x \land x \cdot 0 = 0 \land x \cdot 1 = x ) \] (6)

\[ \forall x \forall y \ ( \ x' = y' \iff x = y ) \] (7)

\[ \forall x \forall y \ ( \ x \leq y \iff (x' \leq y \lor x = y) ) \] (8)

\[ \forall x \forall y \ ( \ x \cdot y' = (x \cdot y) + x \land x + y' = (x + y)' ) \] (9)

\[ \forall x \forall y \forall z \ [ \ x = y \land y = z ] \Rightarrow [ x = z \land z = x ] \] (10)

\[ \forall x \forall y \forall z \ [ \ x = y \land y \leq z ] \Rightarrow x \leq z \] (11)

\[ \forall x \forall y \forall z \ [ \ x = y \land z \leq y ] \Rightarrow z \leq x \] (12)

\[ \forall x \forall y \ ( \ x \leq y \Rightarrow \text{Max}(x, y) = y ) \land ( y \leq x \Rightarrow \text{Max}(x, y) = x ) \] (13)

In the context of the above definition for $Q_0$, the Ax-1 and Ax-2 formalisms will be respectively defined as the union of $Q_0$ with all the instances of the respective induction schemes in Equations (1) and (2), where $\phi(x, y)$ is a $\Delta^R_0$ formula. Similarly, $I\Delta^R_0$ will be defined as the union of $Q_0$ with all instances of Equation (2)'s induction schemas where $\phi(x, y)$ is $\Delta^R_0$.

This paragraph will define a set of $\Pi^R_1$ sentences, called $\text{Trivial-R}$, that has the property that $I\Delta^R_0 + \text{Trivial-R}$ proves the same set of theorems as the more conventional Ax-1 and Ax-2 axiomatization for $I\Sigma_0$. In our discussion, a tuple $(a_0, a_1, a_2, \ldots, a_N)$ is called a $\text{Split Representation}$ of an non-negative integer $x$ when the following condition is satisfied:

\[ x = \sum_{i=1}^{N} a_i \cdot (a_0 + 1)^{i-1} \text{ AND } a_1 \leq a_0 \land a_2 \leq a_0 \land \ldots a_N \leq a_0 \] (14)
For a fixed integer \( N \), let \( \text{Split}^N(x, a_0, a_1, \ldots, a_N) \) denote a \( \Delta_0^R \) formula indicating (14) is satisfied.

For each of the arithmetic operators of \(+\), \(*\), \(\text{Max}\), \(=\), \(\leq\), the axiom system \( \text{Trivial-R} \) will have available a family of \( \Delta_0^R \) predicates and \( \Pi_1^R \) axioms for simulating the operations of these functions when they manipulate \( \text{Split} \) Representations of integers. Thus for a fixed triple \((I, J, K)\), let \( \text{Mult}^{I,J,K}(a_0, a_1, \ldots, a_I, b_0, b_1, \ldots, b_J, c_0, c_1, \ldots, c_K) \) designate a \( \Delta_0^R \) predicate simulating the action of integer multiplication when its input is the two split integers \((a_0, a_1, \ldots, a_I)\) and \((b_0, b_1, \ldots, b_J)\) and its resultant is the multiplicative product of \((c_0, c_1, \ldots, c_K)\). The accompanying \( \Pi_1^R \) axiom of \( \text{Trivial-R} \) that formalizes this predicate will be:

\[
\forall x \forall y \forall z \forall a_0 \forall a_1 \ldots \forall a_I \forall b_0 \forall b_1 \ldots \forall b_J \forall c_0 \forall c_1 \ldots \forall c_K
\[
\{ [ \text{Split}^I(x, a_0, \ldots, a_I) \land \text{Split}^J(y, b_0, \ldots, b_J) \land \text{Split}^K(z, c_0, \ldots, c_K) ] \} \implies
\[
[ x \ast y = z \iff \text{Mult}^{I,J,K}(a_0, a_1, \ldots, a_I, b_0, \ldots, b_J, c_0, \ldots, c_K) ] \}
\]

Likewise, \( \text{Trivial-R} \) will have available analogs of Equation (15) to simulate addition, maximum, equality, and less-than-or-equals among split integers. Thus the predicates of \( \text{Add}^{I,J,K}(a_0, a_1, \ldots, a_I, b_0, b_1, \ldots, b_J, c_0, c_1, \ldots, c_K) \), \( \text{Maxim}^{I,J,K}(a_0, a_1, \ldots, a_I, b_0, b_1, \ldots, b_J, c_0, c_1, \ldots, c_K) \), \( \text{Eq}^{I,J}(a_0, a_1, \ldots, a_I, b_0, b_1, \ldots, b_J) \) and \( \text{LTE}^{I,J}(a_0, a_1, \ldots, a_I, b_0, b_1, \ldots, b_J) \) will be the \( \Delta_0^R \) analogs of \( \text{Mult}^{I,J,K} \) for these four structural relations. Their counterparts of Equation (15)'s formal axiom will then be:

\[
\forall x \forall y \forall z \forall a_0 \forall a_1 \ldots \forall a_I \forall b_0 \forall b_1 \ldots \forall b_J \forall c_0 \forall c_1 \ldots \forall c_K
\[
\{ [ \text{Split}^I(x, a_0, a_I) \land \text{Split}^J(y, b_0, b_J) \land \text{Split}^K(z, c_0, c_K) ] \} \implies
\[
[ x + y = z \iff \text{Add}^{I,J,K}(a_0, a_I, b_0, b_J, c_0, c_K) ] \}
\]

\[
\forall x \forall y \forall z \forall a_0 \forall a_1 \ldots \forall a_I \forall b_0 \forall b_1 \ldots \forall b_J \forall c_0 \forall c_1 \ldots \forall c_K
\[
\{ [ \text{Split}^I(x, a_0, a_I) \land \text{Split}^J(y, b_0, b_J) \land \text{Split}^K(z, c_0, c_K) ] \} \implies
\[
[ z = \text{Max}(x, y) \iff \text{Maxim}^{I,J,K}(a_0, a_I, b_0, b_J, c_0, c_K) ] \}
\]
∀ x ∀ y ∀ a_0 ∀ a_1 ... ∀ a_I ∀ b_0 ∀ b_1 ... ∀ b_J [ Split^I(x, a_0...a_I) ∧ Split^J(y, b_0...b_J) ]

⇒ [ x = y ⇔ Eq^I,J(a_0...a_I, b_0...b_J) ] \ (18)

∀ x ∀ y ∀ a_0 ∀ a_1 ... ∀ a_I ∀ b_0 ∀ b_1 ... ∀ b_J [ Split^I(x, a_0...a_I) ∧ Split^J(y, b_0...b_J) ]

⇒ [ x ≤ y ⇔ LTE^I,J(a_0...a_I, b_0...b_J) ] \ (19)

Henceforth, \textbf{Ax-3} will denote the axiom system \( I∆^R_0 + \text{Trivial-R} \). Section 3 will prove that Ax-3 proves the same set of theorems as Ax-1 and Ax-2.

**Definition 1:** Let \( \alpha \supseteq \beta \) denote that \( \alpha \)'s set of formal axioms includes all \( \beta \)'s axioms. (This definition of \( \supseteq \) is stronger than the more modest construct that \( \alpha \) proves all \( \beta \)'s theorems.) Also assuming \( \alpha \) denotes a \textit{consistent} axiom system and \( D \) denotes a deductive method, \((\alpha, D)\) will be called a \textbf{Threshold} for the Second Incompleteness Effect iff all consistent extensions \( \alpha^* \supseteq \alpha \) have the property that \( \alpha^* \) is unable to prove the consistency of its proofs using deduction method \( D \). Otherwise, \((\alpha, D)\) will be called an \textbf{Anti-Threshold}. (It means that some consistent \( \alpha^* \supseteq \alpha \) can prove a theorem affirming its own consistency under deduction method \( D \).)

Using Definition 1's notation, the main theorem proven in this article will be that Ax-3 is an anti-threshold for the Herbrandized version of the Second Incompleteness Theorem. This means that there must assuredly exist some \textit{consistent} system \( \alpha^* \supseteq \text{Ax-3} \), where \( \alpha^* \) can prove a theorem affirming its own Herbrand consistency.

\section{Basic Framework and Underlying Intuition}

This section will have two purposes. It will formally prove Ax-3 proves the same set of theorems as Ax-1 and Ax-2, thus establishing that Ax-3 is a permissible formal means for axiomatizing \( IΣ_0 \). It will also provide an intuitive explanation of why Ax-3 is able to evade the Herbrandized version of the Second Incompleteness Theorem (by satisfying Definition 1's “anti-threshold” property for Herbrand consistency).

**Theorem 1** \textit{Each of Ax-1, Ax-2 and Ax-3 prove the same set of theorems.}
Proof Sketch: It is well known Ax-1 and Ax-2 prove the same set of theorems. Thus to establish Theorem 1, we need only show Ax-2 and Ax-3 also prove the same set of theorems.

Our proof will use the fact that Paris and Dimitracopoulos [26] have observed that in a model-theoretic sense, there is a 1-to-1 correspondence between $\Delta_0$ formulae and their equivalent representations in a $\Delta_0^R$ form. Let $\psi(x, y)$ denote an arbitrary $\Delta_0^R$ formula. The Paris and Dimitracopoulos [26] result easily implies that for any integer $k$, it is possible to construct a $\Delta_0^R$ formula $\psi^*(x, y_0, y_1, \ldots, y_k)$ that is its logical counterpart of $\psi(x, y)$ under split representations for an integer. More precisely, it implies that one can map $\psi(x, y)$ onto a $\Delta_0^R$ formula $\psi^*(x, y_0, y_1, \ldots, y_k)$ such that the Ax-2 and Ax-3 representations of $I\Sigma_0$ can both trivially prove the following property:

$$\forall x \ \forall y \ \forall y_0 \ \forall y_1 \ldots \forall y_k \{ \text{Split}^k(y, y_0, y_1, \ldots, y_k) \implies [ \psi(x, y) \iff \psi^*(x, y_0, y_1, \ldots, y_k) ] \} \quad (20)$$

Let $\text{Size}_L(y_0, y_1, \ldots, y_k)$ denote a $\Delta_0^R$ formula indicating that $(y_0, y_1, \ldots, y_k)$ represents an integer $\leq L$. Then it is not hard to show that Ax-3 can use its Trivial-R axioms to prove that the two $\Delta_0$ formulae of $\exists y \leq x^k \psi(x, y)$ and $\forall y \leq x^k \psi(x, y)$ are equivalent to the respective $\Delta_0^R$ formulae of:

$$\exists y_0 \leq x \ \exists y_1 \leq x \ldots \exists y_k \leq x \ \text{Size}_{x^k}(y_0, y_1, \ldots, y_k) \land \psi^*(x, y_0, y_1, \ldots, y_k) \quad (21)$$

$$\forall y_0 \leq x \ \forall y_1 \leq x \ldots \forall y_k \leq x \ \text{Size}_{x^k}(y_0, y_1, \ldots, y_k) \Rightarrow \psi^*(x, y_0, y_1, \ldots, y_k) \quad (22)$$

Thus by essentially applying $n$ iterations of this technique (and its obvious analogs) to any initial $\Delta_0$ formula with $n$ bounded quantifiers, Ax-3 can transform an arbitrary $\Delta_0$ formula into a provably equivalent $\Delta_0^R$ formula. It thus follows that although the Ax-3 system contains technically only instances of Equation (2)'s axiom schema for $\Delta_0^R$ formulae, it nevertheless has an ability to formally prove as theorems all the remaining instances of Ax-2's axiom schema for $\Delta_0$ formulae as well. (In particular if $\phi(x, y)$ is a $\Delta_0$ formula which is not $\Delta_0^R$ and if $\phi^*(x, y)$ is a $\Delta_0^R$ formula equivalent to $\phi(x, y)$, then Ax-3 can prove a theorem verifying the validity of Equation (2)'s axiom scheme for $\phi(x, y)$ by first observing that this axiom scheme is valid for $\phi^*(x, y)$ and then observing the latter is equivalent to $\phi(x, y)$’s axiom scheme.)

Hence although Ax-3 contains technically only a proper superset of Ax-2’s induction axioms as formal axioms within its own induction schema, it has the ability to formally prove the validity of all of Ax-2’s axioms. The proof in the reverse direction (establishing that Ax-2 can prove all Ax-3’s axioms) is straightforward because Ax-2 can easily prove all Ax-3’s Trivial-R axioms. Hence, Ax-2 and Ax-3 will prove identical sets of theorems. □
Remark 1. Our proof that Ax-3 is an anti-threshold for the Herbrandized version of the Incompleteness Theorem will appear in Section 4. This result is quite surprising because there are of course many more generalizations of Gödel’s Second Incompleteness Theorem available in the mathematical literature [1–4,6–9,15,19,23,24,28,30–36,38,41,40,43,44,46,48,50,52,54] than there are published examples of boundary-case style exceptions to its formalism.

In order to explain intuitively the core idea about why our paper is able to achieve this surprising evasion of the Second Incompleteness Theorem, it is helpful to let $\Upsilon_n$ denote Equation (23)’s $\Delta_0$ sentence. Note that this sentence is comprised of $O(n)$ logic symbols. It asserts that the variables $v_0, v_1, v_2, \ldots, v_n$, satisfy $v_i = 2^{2^i}$.

$$\exists v_0 \leq 2 \exists v_1 \leq v_0 * v_0 \exists v_2 \leq v_1 * v_1 \ldots \exists v_n \leq v_{n-1} * v_{n-1}$$

$$v_0 = 2 \land v_1 = v_0 * v_0 \land v_2 = v_1 * v_1 \land \ldots \land v_n = v_{n-1} * v_{n-1} \quad (23)$$

It is easy to see there exists some $\Delta^R_0$ sentence, called say $\Upsilon^R_n$, that is the counterpart of Equation (23) written in a notation using split integers. This sentence will indicate the existence of a sequence of split integers $S_0, S_1, S_2, \ldots, S_n$, where $S_i$ represents the quantity $2^{2^i}$.

However although they in some sense represent equivalent concepts, there is a fundamental difference between the $\Delta_0$ sentence $\Upsilon_n$ and its $\Delta^R_0$ counterpart $\Upsilon^R_n$. This difference is easiest to explain if one uses a logical language that has only 3 named constants, 0, 1 and 2, and if split integers are encoded as base 2 numbers. Then $\Upsilon^R_n$ will be encoded as a sequence of at least $2^n$ characters, but $\Upsilon_n$’s length has a sharply smaller $O(n)$ magnitude. As a consequence of this distinction (and its generalizations), we realized that one could formulate an axiom system that was logically equivalent to the more conventional encodings for $I\Sigma_0$, but which was an anti-threshold to the Herbrandized version of the Second Incompleteness Theorem (using Definition 1’s terminology).

In particular, it turns out that the exponential difference between the lengths of the $\Delta_0$ sentence $\Upsilon_n$ and its $\Delta^R_0$ counterpart $\Upsilon^R_n$ plays a major role in understanding the central concept that motivated much of the research that stimulated the current article. Thus, this paradigm (and its formal generalization for arbitrarily more complicated sequences of sentences) are used by Section 4’s machinery in a much more sophisticated context to prove that it is feasible to construct awkward encodings for $I\Sigma_0$, similar to Ax-3, that are actual anti-thresholds to the Herbrandized version of the Second Incompleteness Theorem. (A reader should not worry if he does not follow fully the intuitive idea, sketched in this paragraph, about the significance of logically equivalent statements that have exponentially different sizes in length. This is because
a fully formalized proof of our main theorem, that uses this effect, will be explored in great detail in the next section.)

4 Main Analysis

A sentence $\psi$ in the propositional calculus will be called an **Anti-Tautology** iff $\psi$ is unsatisfiable (i.e. $\neg \psi$ is a tautology). Our definition of Herbrand deduction will be identical to the definitions used by Adamowicz, Hájek-Pudlák and Kołodziejczyk [1,13,19], except that we will use a dual version of this definition that follows from De Morgan’s Rule, where disjunctions are replaced with conjunctions and where tautologies are replaced with anti-tautologies. In other words, our definition will use the well-known identity that

$$\bigvee_{i=1}^{n} \neg \phi_i = \neg \bigwedge_{i=1}^{n} \phi_i$$  \hspace{1cm} (24)

Our definition of Herbrand deduction will differ from its more conventional definitions by using the right (instead of left) side of (24)’s identity. This change in notation is unnecessary, but it does help to considerably simplify our proofs.

Let $\Psi$ denote an arbitrary prenex normal sentence such as the prototype below, whose open subcomponent is denoted as $\tilde{\psi}$.

$$\forall x_1 \exists y_1 \ \forall x_2 \exists y_2 \ ... \ \forall x_n \exists y_n \ \tilde{\psi}(x_1, y_1...x_n, y_n)$$ \hspace{1cm} (25)

In a context where $f_1^{\psi}(x_1)$, $f_2^{\psi}(x_1, x_2)$ ... $f_n^{\psi}(x_1, x_2, \ldots, x_n)$ are new function symbols, Equation (26) is called the Skolemization of Equation (25).

$$\forall x_1 \forall x_2 ... \forall x_n \ \tilde{\psi} \ [ \ x_1, f_1^{\psi}(x_1), x_2, f_2^{\psi}(x_1, x_2) ... x_n, f_n^{\psi}(x_1, x_2, \ldots, x_n) \ ]$$ \hspace{1cm} (26)

In a context where $L$ is a logical language and $\alpha$ is an axiom system, we will let $C_L$ and $F_L$ denote the set of constant and function symbols associated with $L$. Similarly, $F_\alpha$ will denote the set of “Skolemized” function symbols associated with $\alpha$’s axioms. Thus using (25) and (26)’s notation, let $\alpha$ denote a system of axioms $\Psi_1$, $\Psi_2$, $\Psi_3$ ... , and for an arbitrary index $i$ let its Skolemized function symbols carry names such as $f_1^{\psi_1}$, $f_2^{\psi_2}$, $f_i^{\psi_3}$, ... The **Herbrandized Terms** for this ordered pair $(\alpha, L)$ will then be defined to be the set of all terms generated by the constants from the set $C_L$ combined with the functional operations from the set $F_\alpha \cup F_L$. 

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A Herbrandized Instance of a Skolemized axiom is a sentence identical to this axiom except that all its universally quantified variables are replaced by Herbrandized terms. For instance in a context where $T_1, T_2, T_3,...$ are Herbrandized terms, Equation (27) is such an instance of (26)’s axiom:

$$\overline{\psi} [ T_1, f_1^\psi(T_1), T_2, f_2^\psi(T_1, T_2) \ldots, T_n, f_n^\psi(T_1, T_2, \ldots, T_n) ]$$ (27)

Let ⊥ denote the logical constant of FALSE. A Herbrandized Proof of ⊥ from the axiom system $\alpha$ is defined as a finite collection of Herbrandized instances of $\alpha$ together with a proof, in the pure propositional calculus, that the conjunction of these instances is an anti-tautology.

**Definition 2:** Using our revised notation convention, the theorem $\Upsilon$ will be said to have a Herbrandized Proof from the axiom system $\beta$ if and only if the union of the axiom system $\beta$ with the added sentence $\neg \Upsilon$ produces a Herbrandized proof of ⊥.

**More Notation:** Let us say that a function $G(x_1, x_2, \ldots, x_n)$ is a Non-Growth Function iff $G(x_1, x_2, \ldots, x_n) \leq \text{Max}(x_1, x_2, \ldots, x_n)$. Define a set $S$ of functions to be an Arithmetic Controlled Set iff $S$ includes the arithmetic functions of addition, multiplication and successor and all its other functions are non-growth functions. Also, define a term $t$ to be an Arithmetically Controlled Term iff $t$ is a term that uses only the symbols of 0, 1 and 2 as its inputted constants and all its function symbols come from some Arithmetic Controlled Set $S$. Thus if $G_1$ and $G_2$ are non-growth functions, Equation (28) represents an arithmetically controlled term.

$$G_1[ (2 + 1) \ast (1 + 1), 1 + 2 ] \ast G_2( 2 + 2 + 0, 2 + 2 + 1 + 0 )$$ (28)

Also, in a context where $C_t$ and $F_t$ denote the number of constant and function symbols in $t$, we will use the following notation:

1. MinG($t$) will denote the quantity $2^{C_t+F_t}$.
2. Val($t$) will denote the quantity represented by the term $t$.

For example if $G_1(x, y) = |x - y|$ and $G_2(x, y) = \text{Min}(x, y)$ then Equation (28)’s term $t$ will have $\text{Val}(t) = 3 \times 4 = 12$ and $\text{MinG}(t) = 2^{25}$ (because $t$ contains 12 function symbols and 13 constant symbols).

**Lemma 1** Let $t$ be an arithmetically controlled term which satisfies the inequality $\text{Val}(t) \geq 4$. Then $\text{Val}(t) < \text{MinG}(t)$.

**Proof Sketch:** Suppose for some $k \geq 2$, that $\text{Val}(t) = 2^k$. Then it easy to see that $t$’s maximally compressed representation as an arithmetically...
controlled term is “ 2∗2∗...∗2”. Thus MinG(t) = 2^{2k−1} > Val(t) = 2^k is valid in this case because the preceding product has k appearances of the constant 2 connected by k − 1 appearances of the multiplication symbol. Moreover, it is easily proven that terms, which are not powers of 2, are never represented in a more compressed form than the greatest power of 2 that they exceed. Thus Lemma 1 is valid for all terms where Val(t) ≥ 4. □

**Definition 3.** For a fixed constant B > 0, a set S of functions is defined to be a **B−Bounded Arithmetic Set** iff S includes the arithmetic functions of addition, multiplication and successor and all its other functions G satisfy the constraint that

\[ G(x_1, x_2, \ldots, x_n) \leq \text{Max}(x_1, x_2, \ldots, x_n) \text{ when Max}(x_1, x_2, \ldots, x_n) < B \]  

(29)

Also, we will say a term t is a **B-Bounded Arithmetic Term** iff t is a term that uses only the symbols of 0, 1 and 2 as its inputted constants and all its function symbols come from some B-Bounded Arithmetic Set S. Lemma 2 provides the generalization of Lemma 1 for B-bounded arithmetic terms. Its proof is omitted because it is an easy generalization (see footnote 2) of Lemma 1’s proof.

**Lemma 2** Suppose that t is a B−bounded arithmetic term with MinG(t) < B and Val(t) ≥ 4. Then Val(t) < MinG(t).

**Definition 4.** Let Φ denote the Π^R_1 sentence below whose ∆^R_0 subformula is defined by \( \bar{\phi}(a_1, a_2 \ldots a_n) : \)

\[ \forall a_1 \forall a_2 \ldots \forall a_n \bar{\phi}(a_1, a_2 \ldots a_n). \]  

(30)

For any B ≥ 1, Equation (30) is called a **B−Bounded Valid Π^R_1 sentence** iff (31) is valid under the standard model of the natural numbers

\[ \forall a_1 < B \forall a_2 < B \ldots \forall a_n < B \bar{\phi}(a_1, a_2, \ldots, a_n). \]  

(31)

**Definition 5.** An axiom system \( \alpha \) will be said to satisfy the **Canonical Arithmetic Condition** when all \( \alpha \)'s axioms are Π^R_1 sentences and they include \( Q_0 \)'s nine axioms (i.e. Equations (5)–(13)).

\footnote{The intuitive reason that Lemmas 1 and 2 have similar proofs is that arithmetically controlled terms and B−bounded arithmetic terms have precisely identical growth rates until a construction process builds an intermediate object t with MinG(t) ≥ B.}
Definition 6. Let $\Theta$ denote a methodology for assigning Gödel numbers to Herbrand proofs (which are henceforth denoted as $P$). Let us recall that $\text{MinG}(t)$ was defined by Item (1) in this section. Define $\Theta$ to be a **Conventional Encoding Method** if $\Theta(P) > \text{MinG}(t)$ whenever the proof $P$ contains the Herbrand term $t$. (Such encodings are called “conventional” because all usual methods for encoding Herbrand proofs satisfy $\Theta(P) > \text{MinG}(t)$.)

**Theorem 2** Suppose $\alpha$ is a canonical arithmetic axiom system consisting of $B$-Bounded Valid $\Pi^R_1$ sentences and $\Theta$ again satisfies Definition 6’s Conventional Encoding property. Then any Herbrand proof $P$ of $\bot$ from the axiom system $\alpha$ will satisfy the inequality that $\Theta(P) > B$.

**General Comments about Theorem 2 and its Proof:** At an intuitive level, Theorem 2 can be viewed as a consequence of the machineries of Lemma 2 and Definitions 3-6. This is because the B-Bounded validity condition in Theorem 2’s hypothesis can be used to show that a Herbrand proof $P$ of $\bot$ must contain some term $t$ where $\text{Val}(t) \geq B$. In this context, the combination of Lemma 2 and Definition 6 will imply that such a term will force $P$’s Gödel number to exceed the lower bound of $B$.

A more detailed formal proof of Theorem 2 appears in Appendix A. It explains the precise role that Definition 3 and Lemma 2 play in establishing this theorem. Our recommendation is that a reader postpone examining Appendix A until after he finishes the remainder of this section. It will explain the significance of Theorem 2 by showing how it enables us to prove the surprising result that the Ax-3 axiomatization for $I\Sigma_0$ is an anti-threshold for the Herbrandized version of the Second Incompleteness Theorem.

**Theorem 3** For any arbitrary axiom system $\alpha$ and deduction method $D$, let $\text{Diagonal}(\alpha, D)$ and $\alpha^D$ denote the following two constructs:

**A.** $\text{Diagonal}(\alpha, D)$ will denote a logical sentence that states: “There is no proof (using deduction method $D$) of the falsity sentence $\bot$ from the union of the axiom system $\alpha$ with this sentence $\text{Diagonal}(\alpha, D)$ (looking at itself).”

**B.** $\alpha^D$ will denote the formal union of the axiom system $\alpha$ with the sentence $\text{Diagonal}(\alpha, D)$.

Let $\text{Diag}(Ax-3)$ denote the special variant of $\text{Diagonal}(\alpha, D)$ where $\alpha = Ax-3$ and $D$ designates Herbrand deduction. Both these two constructs and also $\alpha^D$ are well defined. Also, $\text{Diag}(Ax-3)$ has a $\Pi^R_1$ encoding.
Abbreviated Sketch of Theorem 3’s proof. As early as 1938, Kleene observed [17] that the Fixed Point Theorem implied that a type of cousin of the sentence Diagonal(α, D) was well defined. More recently, Willard [49,52] showed how Diagonal(α, D) could be formally endowed with a Π_R^1 encoding under the conventional language of arithmetic. It is reasonably straightforward to generalize [49,52]’s result to establish that Diag(Ax-3) also has a well defined Π_R^1 encoding (thus completing Theorem 3’s proof.) The remainder of this proof sketch will summarize the ideas from [49,52] for the benefit of those readers who are unfamiliar with this topic. Our discussion will employ the following notation:

i) Prf^D_α (t, p) will denote a formula designating that p is a proof of the theorem t from the axiom system α using the deduction method D.

ii) ExPrf^D_α (h, t, p) will be a formula stating that p is a proof (using the deduction method D) of a theorem t from the union of the axiom system α with the added axiom sentence whose Gödel number equals h.

iii) Subst (g, h) will denote Gödel’s classic substitution formula — which yields TRUE when g is an encoding of a formula and h is an encoding of a sentence that replaces all occurrence of free variables in g with a mathematical term formalizing the Gödel number for representing “g”.

iv) SubstPrf^D_α (g, t, p) will denote the natural hybridizations of the constructs from Items (ii) and (iii) which yields a Boolean value of TRUE exactly when there exists some integer h simultaneously satisfying both Subst (g, h) and ExPrf^D_α (h, t, p).

Each of (i)–(iv) can be encoded as ∆^R_0 formulae when α is any recursively enumerable axiom system. In particular, Appendices C and D of [49] essentially established (see footnote 3) that the first three of these predicates can receive ∆^R_0 encodings when one applies the theory of LinH functions from [13,21,56] in a reasonably routine manner. In such a context, Equation (32) illustrates one possible ∆^R_0 encoding for SubstPrf^D_α (g, t, p)’s graph. (It is equivalent to “ ∃ h [ Subst(g, h) ∧ ExPrf^D_α (h, t, p) ] ”, but Equation (32) is a ∆^R_0 formula — unlike the quoted expression.)

\[ \text{Prf}^D_\alpha (t, p) \lor \exists h \leq p [ \text{Subst} (g, h) \land \text{ExPrf}^D_\alpha (h, t, p) ] \]  (32)

3 The results of Wrathall [56] have been noted by Hájek–Pudlák [13] to imply that every LinH function [13,21,56] has a ∆_0 encoding. Using a slightly different “ ∆_0 ” notation, the results from Appendices C and D of [49] explained how this result would imply that the each of Prf^D_α (t, p), ExPrf^D_α (h, t, p) and SubstPrf^D_α (g, t, p) have “ ∆_0 ” encodings. Since the ∆^R_0 class of formulae is broader than ∆_0, these formulae must also have ∆^R_0 encodings.
Utilizing (32)'s $\Delta^R_0$ encoding for $\text{SubstPrf}^D_\alpha(g, t, p)$, it is easy to formulate a $\Pi^R_1$ encoding for the axiom sentence $\text{Diagonal}(\alpha, D)$. Thus, let $\Gamma(g)$ denote Equation (33)'s formula, and let $N$ denote $\Gamma(g)$'s Gödel number. Then $\Gamma(N)$ is a $\Pi^R_1$ encoding for $\text{Diagonal}(\alpha, D)$.

\[ \forall p \neg \text{SubstPrf}^D_\alpha(g, \bot, p) \quad (33) \]

\[ \square \]

**Clarifying Comment:** One should be somewhat cautious in interpreting the meaning of Theorem 3. It does not indicate that $\text{Diag}(\text{Ax-3})$ is a logically valid statement under the standard model of the natural numbers. Rather, it merely indicates $\text{Diag}(\text{Ax-3})$ is a well defined $\Pi^R_1$ sentence. In order to establish prove that $\text{Diag}(\text{Ax-3})$ is also valid, we will need the added force of Theorem 4 below.

**Theorem 4** Let $\text{Ax-3}^*$ denote the union of the axiom system $\text{Ax-3}$ with the sentence $\text{Diag}(\text{Ax-3})$. Then $\text{Ax-3}^*$ is consistent. (Thus, $\text{Ax-3}$ is an “anti-threshold” for the Herbrandized version of the Second Incompleteness Theorem under Definition 1’s notation convention.)

**Proof of the Consistency Property of $\text{Ax-3}^*$ :** Suppose for the sake of establishing a proof-by-contradiction that $\text{Ax-3}^*$ was inconsistent. Then one could identify a proof $P$ of $\bot$ whose Gödel number $\Theta(P)$ is the smallest Gödel number of a Herbrand proof of $\bot$ from $\text{Ax-3}^*$. We will now construct from $P$ an alternate Herbrand proof $R$ of $\bot$ where $\Theta(R) < \Theta(P)$. The formal construction of such a $R$ will suffice for our proof by contradiction to reach its desired end because such a $R$ cannot possibly exist (on account of $P$’s minimality property).

Our strategy is to use Theorem 2 to construct $R$ from $P$. Theorem 2 is relevant to $\text{Ax-3}^*$ because all the formal axiom sentences of $\text{Ax-3}^*$ are assuredly $\Pi^R_1$ sentences (see footnote 4 for the justification of this claim). In such a context, we may apply Theorem 2 to conclude that for some $B < \Theta(P)$, at least one of the axiom sentences of $\text{Ax-3}^*$ must fail to be a $B$-Bounded valid $\Pi^R_1$ sentence. Moreover, it is obvious that all the axioms of $\text{Ax-3}$ possess an unbounded level of validity (i.e they are $B$-Bounded valid for all possible $B$). Hence, these two observations imply $\text{Diag}(\text{Ax-3})$ fails to be $B$-bounded valid (simply because some axiom from $\text{Ax-3}^*$ must fail to be $B$-bounded valid, and $\text{Diag}(\text{Ax-3})$ is the only available axiom belonging to $\text{Ax-3}^*$ that is not also a member of $\text{Ax-3}$.)

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4. Theorem 3 implies that the $\text{Diag}(\text{Ax-3})$ axiom of $\text{Ax-3}^*$ has a $\Pi^R_1$ structure, and Section 2’s definition of $\text{Ax-3}$ implies that all the other axiom-sentences belonging to $\text{Ax-3}$ are certainly also $\Pi^R_1$. 

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The latter observation, combined with Diag(Ax-3)’s definition implies (see footnote 5) that some \( R \) with \( \Theta(R) < B \) must be another proof of \( \perp \). Hence our last two inequalities certainly imply that \( \Theta(R) < B < \Theta(P) \). This finishes our proof-by-contradiction because \( P \)'s previously presumed minimality has been contradicted by \( R \). □

**Remark 2.** Our discussion in this section had assumed that the terms \( T \) in a \( \Delta^0 \)’s formula’s bounded quantifiers included only the Maximum function symbol. The results of Theorem 4 would actually also hold if these quantifiers were also permitted to include the Addition function symbol. (The only reason our discussion had omitted the possibility that both the addition and maximum function symbols appear in the \( \Delta^0 \) formula \( \phi \)’s bounded quantifiers in Equation (2) was for the sake of simplifying the presentation.)

**Remark 3.** The attached Appendix B discusses a yet further reason why Theorem 4 is of interest. The anonymous referee had suggested we add this appendix to the current paper. Its methodologies are related to Kołodziejczyk’s observation [18,19] that semantic tableaux and Herbrand deduction can sometimes have an exponential difference in their proof lengths. The purpose of Appendix B is to sketch how one can generalize [55]’s results for Ax-1 and Ax-2 to establish that Ax-3 is also a threshold for the semantic tableaux version of the Second Incompleteness Theorem. In a context where Theorem 4 had established the polar opposite result for Ax-3 under Herbrand deduction, this contrast is, of course, quite interesting.

### 5 Discussion of Significance of Results

A comparison between our research and the prior research of Kreisel-Takeuti and Pudlák [22,31,39] was postponed until the closing part of this current article because the results from Sections 3 and 4 were needed to precede this discussion.

In this section, \( S(x) \) will denote the “successor” operation that maps the integer \( x \) onto \( x + 1 \). A formula \( \varphi(x) \) is called [13] a **Definable Cut for** an axiom system \( \alpha \) iff \( \alpha \) can prove:

\[
\varphi(0) \land \forall x \{ \varphi(x) \Rightarrow \varphi[S(x)] \} \land \forall x \forall y < x \{ \varphi(x) \Rightarrow \varphi(y) \} \quad (34)
\]

The strictly formalistic definition of Diag(Ax-3) as the entity “ \( \Gamma(N) \) ” (using a self-reference principle) can be found in the last sentence of Theorem 3’s proof. The syntax of its Equation (33) implies that if Diag(Ax-3) fails to be \( B \)–Bounded valid then another proof \( R \) must assuredly exist for the sentence \( \perp \) with \( \Theta(R) < B \).

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5 The strictly formalistic definition of Diag(Ax-3) as the entity “ \( \Gamma(N) \) ” (using a self-reference principle) can be found in the last sentence of Theorem 3’s proof. The syntax of its Equation (33) implies that if Diag(Ax-3) fails to be \( B \)–Bounded valid then another proof \( R \) must assuredly exist for the sentence \( \perp \) with \( \Theta(R) < B \).
A very extensive literature [1,3,6,10,12,13,16,20,21,25,27–33,35,37,42–46,50,51] has studied Definable cuts. We have published two 6-page summaries of this literature in the review chapters of our articles [52,54]. (Also, Pudlak’s full-length survey article [32] offers an excellent review of the work done in this subject prior to approximately 1990.)

All axioms systems, strictly weaker than Peano Arithmetic, contain some definable cut that is not provably equivalent to the full set of integers. In the proof-theory literature, the definition of a “Definable Cut” (see [13]) is formally unrelated to Gentzen’s notion of a sequent calculus deductive “cut rule”, despite their very similar sounding names.

Let \( \lceil \Psi \rceil \) denote \( \Psi \)'s Gödel number, and \( \text{Prf}^D_\alpha(t,p) \) denote that \( p \) is a proof of the theorem \( t \) from the axiom system \( \alpha \) using the deduction method \( D \). Also, let \( \varphi(x) \) denote a definable cut. Let us say that an axiom system \( \alpha \) can recognize its Cut-Localized \( D \)-consistency under \( \varphi(x) \) iff \( \alpha \) can formally prove:

\[
\forall p \{ \varphi(p) \Rightarrow \neg \text{Prf}^D_\alpha(\lceil 0 = 1 \rceil, p) \}
\] (35)

Pudlák [31] proved that no consistent extension of the Tarski-Mostowski-Robinson [41] system \( Q \) can prove the validity of Equation (35) for any definable cut when \( D \) denotes either Hilbert deduction or a Gentzen sequent calculus system with a deductive cut rule.

At the same time, the literature about Definable Cuts has also defined some circumstances where (35) is provable from \( \alpha \) when for example \( D \) denotes either semantic tableaux or Herbrand deduction. The strongest result of this type was discovered by Pudlák [31]. For both Herbrand and semantic tableaux deduction, [31] showed that it is possible to construct a definable cut \( \varphi(p) \) where \( \alpha \) can prove the validity of Equation (35)'s Cut-Localized \( D \)-consistency property when \( \alpha \) is an axiom system with finite cardinality that satisfies a relatively minor additional constraint called “sequentiality”.

A second class of boundary-like exceptions to the Second Incompleteness can be constructed via the “Cut Free Analysis” (CFA) systems of Kreisel and Takeuti [22,39]. It has consisted of a Second Order Logic generalization of the cut-free version of Gentzen’s Sequent Calculus that is able to prove some types of versions of statements about its own consistency.

The work of Kreisel and Takeuti [22,39] had chronologically preceded most of the literature about Definable Cuts. There is a fascinating partial analogy between the perspectives of these two approaches. This is because the Kreisel-Takeuti papers [22,39] can be viewed as using an analog of Equation (35) in an implicit manner. Thus [22] employs a set of objects, which we shall call \( I \),
that includes all the standard integers plus plausibly non-standard integers that can represent a proof of a contradiction. It then uses $I$ to construct a subset of it, called "$N$", which can be viewed as a representation of the natural numbers. (More precisely, the pages 16-17 of [22] construct $N$ from $I$ in this manner by employing Dedekind’s and Zermelo’s inductive second-order logic definitions of the natural numbers. They thus formally treat the Dedekind and Zermelo inductive definitions of $N$ as an analog of $\varphi(p)$ in Equation (35)’s Cut-Localized D-Consistency statement.)

A question which then naturally arises is whether or not one can also develop some types of axiom systems which can prove their own consistency without relying upon Equation (35)’s “Cut-Localized D-consistency” machinery (or [22]’s analog of it for a second-order generalization of a Gentzen Cut-Free logic.) Such an approach is desirable because one would ideally like to characterize the properties of the natural numbers directly — instead of being required to view them as a subset of a potentially larger set of objects, called $I$.

It is indeed possible to make some progress in this area by using a third approach, relying upon the Diagonal($\alpha, D$) sentence, which was formally defined by the Theorem 3. (Analogs of this Diagonal($\alpha, D$) sentence have also been used in several of our previous papers [47,49,52,54] in some contexts quite different from the paradigm outlined by Theorem 4.)

It is difficult to make more detailed comparisons between the partial exceptions to Gödel’s Second Theorem that use our Diagonal($\alpha, D$) sentence with the methods of Kreisel-Takeuti and Pudlák [22,31,39]. Each technique has its own distinct separate advantage. There also appears to be no natural way to hybridize these techniques. (The point is that each of these different methodologies is looking at a different type of problem setting, where a different form of solution method is available.)

In particular, our research has treated the sentence Diagonal($\alpha, D$) as an axiom of $\alpha$ while the Kreisel-Takeuti and Pudlák papers [22,31,39] treat their analogs of Equation (35) as derived theorems. In this context, it is very natural for a reader to inquire whether is is preferable to treat an “I am consistent” statement as a theorem rather than as an axiom of the system $\alpha$ that generates it?

There is surprisingly no easy answer to this question. For instance, if one’s goal is to attempt to return to the original objectives of Hilbert’s consistency program, then the Kreisel-Takeuti and Pudlák approaches are quite significant because they indicate some respects in which an axiom system can indeed formally prove at least a reduced versions of its consistency statement. However from an alternate perspective, there is a substantial difficulty with approaches
that treat the statement “I am consistent” and its analogs as a theorem rather than as an axiom. The difficulty arises when the relevant systems do not have an operating modus ponens or Gentzen-like cut rule (as is the case with the axiom systems of [22,31]). Such systems cannot draw inferences when they prove a theorem essentially declaring that “I am consistent”. However, a similar such difficulty does not affect the self-justifying axiom systems in our earlier papers [47,49,52–54] or in Theorem 4 because they treat the statement Diagonal(α, D) as a formal axiom rather than as a theorem. (The point here is that axiom sentences, quite unlike theorem sentences, can be permissibly used as intermediate steps to generate other deductions under the inference rules of semantic tableaux, the cut-free sequent calculus and/or Herbrand deduction — thus causing their presence to have stronger implications.)

Thus, there are quite different insights that arise from systems that treat variants of the “I am consistent” statement as an axiom instead of as a theorem (because of the different connotations these two approaches carry). Neither approach is uniformly preferable over the other. Another difference between our research and the investigations of Kreisel-Takeuti and Pudlák [22,31,39] is that the sentence Diagonal(α, D) declares the consistency of α in a global sense whereas Equation (35) refers only to the localized subset of integers that lie within the domain of ϕ. (In the case of Kreisel-Takeuti second-order logic system, Equation (35) is used implicitly on the pages 16 and 17 of [22] to draw the distinction between what we have called I and N earlier in this section.)

When one compares our results in Theorem 4 or in our earlier papers [47,49,52–54] with the research of Kreisel-Takeuti and Pudlák [22,31,39], it is best to remember that Gödel’s Second Incompleteness Theorem precludes an exception to it from becoming too powerful. Thus, it is possible to develop different types of partial evasions to the Second Incompleteness Theorem around its periphery that shed different types of useful new perspectives. However, it is awkward to attempt to hybridize these different types of partial evasions into a stronger uniform evasion. This is because if such a hybrid combined the strengths of the different approaches X, Y and Z, then it would become so strong that its properties could potentially violate the statement of Gödel’s historic result. Thus, there are trade-offs where different types of useful insights, stemming from different approaches (that are difficult to hybridize into one unified methodology), examine somewhat different problems and produce different new perspectives.

Part of the reason that Ax-3’s evasion of the Herbrandized version of the Second Incompleteness Theorem is interesting is that it is known that further improvements upon this result are very difficult to obtain. Thus, Adamowicz-Zbierski [1,3] showed that Σ₀<sub>0</sub>+Ω₁ was unable to verify its Herbrand consistency, Willard [48,50,55] modified this result to show the Second Incompleteness Theorem applied also to Ax-1 and Ax-2’s versions of Σ₀<sub>0</sub>’s axiomatization
under semantic tableaux deduction, and Salehi [33] extended some of the earlier results from [1,3,48,50] to develop some further incompleteness results for $\Sigma_0$ under Herbrand deduction. Thus in a context where several earlier papers have explored types of results where the Second Incompleteness Theorem holds for encodings for $\Sigma_0$ and its cousins, it is surprising that the current article has documented a paradigm where at least the Ax-3 encoding for $\Sigma_0$ evades the Herbrandized version of the Second Incompleteness Theorem.

6 Broader Perspectives

Some added notation is useful so that we can examine Theorem 4’s results from a broader perspective. Let us say a formula $\Phi$ is $\Sigma^R_1$ when it can be written in the form $\neg \Psi$ where $\Psi$ is $\Pi^R_1$. Some other fairly conventional notation is that a sentence will be called $\Pi^R_{k+1}$ when it can be written in the form $\forall v_1 \forall v_2 \ldots \forall v_n \phi(v_1, v_2, \ldots, v_n)$ where $\phi(v_1, v_2, \ldots, v_n)$ is a $\Sigma^R_k$ formula. Likewise, a sentence will be called $\Sigma^R_{k+1}$ when it can be written as $\exists v_1 \exists v_2 \ldots \exists v_n \phi(v_1, v_2, \ldots, v_n)$ where $\phi(v_1, v_2, \ldots, v_n)$ is a $\Pi^R_k$ formula. Also, a sentence will be said to be Level $-k$ if it is either $\Pi^R_k$ or $\Sigma^R_k$.

Definition 7. Let $H$ denote a sequence of ordered pairs $(t_1, p_1)$, $(t_2, p_2)$, ..., $(t_n, p_n)$, where $p_i$ is a Herbrand proof of the theorem $t_i$. For an arbitrary integer $k \geq 1$, this list $H$ will be defined to be a Herb $-k$ proof of a theorem $T$ from the axiom system $\alpha$ iff $T = t_n$ and also:

1. Each axiom in $p_i$’s proof is either one of $t_1, t_2, \ldots, t_{i-1}$ or comes from $\alpha$.
2. Each of the “intermediately derived theorems” $t_1, t_2, \ldots, t_{n-1}$ must lie within the Level-$k$ class of sentences.

Intuitively, Herb $-k$ deduction can be viewed as an extension of Herbrand deduction that contains a type of Gentzen-like deductive cut rule for Level-$k$ sentences.

The Definition 7’s machinery is useful for helping to describe both the maximal generalizations as well as inherent limitations of Section 4’s formalism. Thus using Definition 7’s notation, one can establish the following two tightly complementary negative and positive results:

I. There exists a logically valid $\Pi^R_1$ sentence denoted as $\Psi$ such that no consistent axiom system can contain $\Psi$ as an axiom and prove a theorem affirming its own consistency under Herb $-2$ deduction.
In contrast for each consistent axiom system $A$, there exists a consistent axiom system $I(A)$ that can prove all $A$’s $\Pi_R^1$ theorems and recognize its own consistency under Herb$–1$ deduction.

In other words, Items I and II indicate that there is a fundamental difference between Herb$–1$ and Herb$–2$ deduction. Thus, Herb$–1$ deduction allows for a type of robust evasion of the Second Incompleteness Theorem under a formalism that contains a type of limited Gentzen-style cut rule. However, Result-II precludes this evasion from becoming too strong.

We will not prove results I and II here because each has a rather long proof. Also, partial analogs of these two prior results have appeared in our prior work. Thus, [51] and [52] have used the term Tab$–k$ deduction to refer to a construct similar to Herb$–k$ deduction except for the following two changes:

1. Under Tab$–k$ deduction, the proofs $p_1, p_2, \ldots, p_n$ associated with the sequence $(t_1, p_1), (t_2, p_2), \ldots, (t_n, p_n)$, are semantic tableaux proofs instead of Herbrand proofs.
2. Also under Tab$–k$ deduction, the intermediate results $t_1, t_2, \ldots, t_{n–1}$ will technically represent what [51,52] had called $\Pi^*_k$ and $\Sigma^*_k$ sentences (instead of being $\Pi_R^k$ and $\Sigma_R^k$ sentences). These $\Pi^*_k$ and $\Sigma^*_k$ sentences can be intuitively viewed as being roughly analogous to $\Pi_R^k$ and $\Sigma_R^k$ sentences — except that they contain no multiplication function symbol. (They instead use a relation primitive $M(x, y, z)$ to treat multiplication as a 3-way relation.)

In essence Item-I’s result can be viewed as being analogous to [51]’s main theorem. Similarly, Item-II’s result can be viewed as following from a rather natural hybridization of [52]’s main result with the machinery that we had developed in Section 4.

For further details about this material, the reader should examine [51,52]’s generic formalisms. The reason we had slightly changed the topic from semantic tableaux to Herbrand deduction in the current paper is because Herbrand deduction and its Herb$–1$ generalization possess one interesting quality that semantic tableaux and Tab$–1$ deduction simply do not possess. This is that the Ax-3 encoding of the axiom system $I\Sigma_0$ is an anti-threshold to the Herbrandized (and also Herb$–1$) versions of the Second Incompleteness Theorem. However, the similar anti-threshold effect does not apply also to semantic tableaux deduction (as is explained in Appendix B). Thus, the Herb$–1$ deduction method will support certain types of evasions of the Second Incompleteness Theorem that have no analogs under Tab$–1$ deduction. (The interested reader should also examine Kołodziejczyk work [18,19], which observed how semantic tableaux and Herbrand-styled proofs can sometimes have an exponential difference in their lengths.)
In essence the over-all goal of our research, in the current paper and in the previous papers [47] – [55], has been to attempt to sharpen the academic community’s understanding of the meaning of the Second Incompleteness Theorem, by exploring both its maximal generalizations and permitted allowed boundary-case exceptions. We plan to prepare a new article about this subject in the near future, describing the underlying philosophical motivation for much of this research. One must clearly approach this subject matter carefully because the many generalizations of the Second Incompleteness Theorem are seemingly more significant than its occasional boundary-case exceptions. The reason that the partial exceptions to the Second Incompleteness Theorem should not be ignored is because Gödel’s Incompleteness Theorem is often regarded as the paramount discovery of 20th century mathematics. It thus beckons the academic community to explore both its maximum generalizations and possible boundary case exceptions, so that an understanding of the full meaning of its historic result can be sharpened and made more precise. Within such a limited-but-precise framework, the anomalous behavior of Ax-3, documented in this article, should be of scholarly interest.

Acknowledgments: I warmly thank Leszek Kołodziejczyk for the interesting questions that he had emailed me on November 16, 2005. Those questions played a major role in encouraging me to re-examine my earlier work in [50] for one more time. I also thank Zofia Adamowicz and Konrad Zdanowksi, who helped at least partially stimulate the emailed questions from L. Kołodziejczyk through their conversations with him in Warsaw. Those questions from Leszek Kołodziejczyk had greatly stimulated my research in this article. I also thank the anonymous referee for his many useful comments, including the suggestion that I add the Appendix B to this article.

Appendix A: The Proof For Theorem 2

This appendix will explain in further detail how Definitions 3-6 and Lemma 2 may be used to prove Theorem 2. Our discussion will begin with one further definition and two further lemmas.

Definition 8. Consider the possibility that $\Psi$ is the prenex normal sentence, whose open part is formalized by $\tilde{\psi}(x_1, y_1, ...x_n, y_n)$, shown in Equation (36) and whose Skolemized normalized form is illustrated by Equation (37).

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots \forall x_n \exists y_n \tilde{\psi}(x_1, y_1, ...x_n, y_n) \quad (36)$$

$$\forall x_1 \forall x_2 \cdots \forall x_n \tilde{\psi}[x_1, f_1^{\psi}(x_1), x_2, f_2^{\psi}(x_1, x_2), ... x_n, f_n^{\psi}(x_1, x_2, ..., x_n)] \quad (37)$$
For any $B \geq 1$, Equations (36) and (37) will be called a $B$–Bounded Good Skolemization iff one can define (37)’s Skolem functions $f_1^\psi, f_2^\psi, \ldots, f_n^\psi$ so they satisfy both Definition 3’s $B$–Bounded requirement (see footnote 6) and Equation (38) under the standard model of the natural numbers.

$$\forall x_1 < B \forall x_2 < B \ldots \forall x_n < B$$

$$\tilde{\psi} [ x_1, f_1^\psi(x_1), x_2, f_2^\psi(x_1, x_2) \ldots, x_n, f_n^\psi(x_1, x_2 \ldots, x_n) ] \quad (38)$$

Likewise, we will say an axiom system $\alpha$ has a $B$–Bounded Good Skolemization iff all its axioms are so Skolemized.

**Lemma 3** Using the notation conventions from Definitions 4 and 8, every $B$–Bounded Valid $\Pi^R_1$ sentence can be rewritten into a logically equivalent form that has a $B$–Bounded Good Skolemization.

**Proof.** Follows immediately from the definitions of Bounded Validity and Bounded Good Skolemizations (i.e. see Definitions 4 and 8).

**Remark 4:** From Lemma 3, one can gain a further intuitive appreciation of the role that Definition 3 will play in our proof of Theorem 2. This lemma indicated that every $B$–Bounded Valid $\Pi^R_1$ sentence had a $B$–Bounded Good Skolemization, and Definition 8 indicated that such skolemizations satisfied Definition 3’s requirement that no such invoked Skolem function would grow faster than the multiplication primitive. Such slow-growth Skolem functions characterize Ax-3 (but not also the Ax-1 and Ax-2 systems). The intuitive reason for this distinction is that the paradigm in the footnote 6 of Definition 8 applies only to Ax-3.

**Lemma 4** Using the notation conventions from Definitions 5, 6 and 8, suppose that $\alpha$ is a canonical arithmetic system consisting of prenex sentences with $B$–Bounded Good Skolemizations and that $\Theta$ satisfies the Conventional Encoding property. Then any Herbrand proof $P$ of $\bot$ from the axiom system $\alpha$ will satisfy $\Theta(P) > B$.

**Proof-by-contradiction:** Consider the contrary possibility that the inequality $\Theta(P) > B$ failed and that $P$ is a Herbrand-proof of $\bot$ from the system $\alpha$ where $\Theta(P) \leq B$. We shall denote this inequality as $\ast \ast \ast \ldots$.

---

6 The function of Definition 3’s $B$–Bounded requirement is that it assures that each of Equation (37)’s Skolem functions of $f_1^\psi, f_2^\psi, \ldots$ satisfy the constraint that $f_i^\psi(x_1, x_2, \ldots, x_i) \leq \text{Max}(x_1, x_2, \ldots, x_i)$ whenever $\text{Max}(x_1, x_2, \ldots, x_n) < B$. 

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Definition 6 had indicated every term $T$ in the proof $P$ satisfies the inequality of $\Theta(P) > \text{MinG}(T)$. Also, Lemma 2 implied $\text{Val}(T) < \text{MinG}(T)$. These inequalities and *** imply that every term $T$ in the proof $P$ satisfies

$$\text{Val}(T) < B$$

Equation (39) implies all the terms $T_1, T_2, T_3...$ in the Herbrandized instances in the proof $P$ satisfy $\text{Val}(T_i) < B$. The normalized form of an instance of a Skolemized axiom is illustrated by Equation (40). The combination of our $\text{Val}(T_i) < B$ inequalities together with (38)’s $B$–Bounded constraint on $\alpha$’s axioms implies that each such instance of (40) appearing in the proof $P$ must be automatically valid under the standard model of the natural numbers.

$$\tilde{\psi} [ T_1, f_1^\psi(T_1), T_2, f_2^\psi(T_1,T_2), \ldots, T_n, f_n^\psi(T_1,T_2,\ldots,T_n) ]$$

The latter observation completes our proof-by-contradiction because it contradicts the statement *** that had started our proof. More precisely *** had asserted that $P$ was a Herbrand-proof of $\bot$ from the axiom system $\alpha$. However, the Footnote 7 shows that such is impossible when the last sentence of the preceding paragraph indicated that each instance of (40)’s Skolemized axiom is valid under the standard model of the natural numbers. □

**Finishing the Proof for Theorem 2.** It is easy to combine the machineries of Lemmas 3 and 4 to complete the proof of Theorem 2. This is because Lemma 3 had indicated that every $B$–Bounded Valid $\Pi_1^R$ sentence can be rewritten into a logically equivalent form that has a $B$–Bounded Good Skolemization. Thus, Theorem 2 follows by simply taking such rewritten forms of $\alpha$’s axioms and then applying Lemma 4’s machinery. □

**Appendix B: An Analysis of Ax-3’s Semantic Tableaux Properties**

This appendix will illustrate how the methods of [55] may be extended to prove that Ax-3 satisfies the semantic tableaux version of the Second Incompleteness Theorem. Our discussion will be closely related to Kołodziejczyk’s observations [18,19] that (in the context of Buss’s Bounded Arithmetic [5])

7 The point here is simply that a conjunction of Skolemized instances can produce a proof of $\bot$ only when there exists no model $M$ where all these instances are simultaneously valid. Hence when the preceding paragraph shows that all these Skolemized instances are simultaneously valid under the Standard Model of the Natural Numbers, it implies that certainly no such proof of $\bot$ can feasibly exist.
semantic tableaux and Herbrand deduction can sometimes have an exponential or greater difference in their proof lengths. In a context where Theorem 4 had showed that Ax-3 was an anti-threshold under Herbrand deduction, the results in this appendix are noteworthy because they imply Ax-3 has polar opposite qualities under semantic tableaux and Herbrand deduction.

The discussion in this abbreviated appendix will assume that the reader is familiar with [55]’s proof that Ax-1 and Ax-2 satisfy the semantic tableaux version of the Second Incompleteness Theorem. We will also often rely upon the notation convention from the second paragraph of Section 2 of [55] (which defined semantic tableaux deduction’s eight elimination rules).

It is desirable to examine a fourth axiomization for $\Sigma_0$, called Ax-4, before considering Ax-3 because such an approach will help make the underlying intuitions behind our methodologies easier to appreciate. In our discussion, the symbol $\Psi$ will denote the $\Pi_1^R$ sentence defined by Equation (41). Ax-4 will be defined to be an encoding of $\Sigma_0$ that is identical to Ax-3 except that it includes Equation (41)’s added sentence. (Since Ax-3 can trivially prove the validity of Equation (41), the Ax-4 system is clearly logically equivalent to Ax-3. Thus while proof lengths may differ in these two axiom systems, the final theorems that they derive are the same.)

\[ \forall z \forall q \leq z \ [ \ q * q \leq z \Rightarrow \exists r \leq z ( r = q * q ) ] \]  

**Lemma 5** The axiom system Ax-4 satisfies the semantic tableaux version of the Second Incompleteness Theorem. (In other words using Definition 1’s notation, Ax-4 is a “threshold for the Second Incompleteness Effect” under semantic tableaux deduction).

**Proof Sketch:** Our justification of Lemma 5 will employ the definition of semantic tableaux deduction that had appeared in Section 2 of [55]. Its second paragraph listed eight elimination rules for semantic tableaux deduction. For any term $s$, its sixth rule applies to bounded existential quantifiers appearing in expressions similar to:

\[ \exists v \leq s \ \Theta(v) \]  

For an arbitrary new constant symbol $U$ that does not appear in the base axiom system $\alpha$ or in any higher node in the semantic tableaux proof tree, this rule 6 allows Equation (43) to be a descendant of Equation (42)’s node.

\[ U \leq s \ \land \ \Theta(U) \]  

Also consider Equation (44)’s universally quantified sentence.

\[ \forall v \Phi(v) \]  

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In this context for any term \( t \) which is free in \( \Phi \), the seventh elimination rule in Section 2 of [55] indicated that a descendant of Equation (44)’s sentence in a semantic tableaux proof tree is allowed to be any sentence of the form \( \Phi(t) \). In particular if we take \( t \) to be a term of the form “\( U \ast U \)” (where \( U \) was defined in Equation (43) ) then this universal quantifier elimination rule may produce the following reduction from Equation (44).

\[
\Phi(U \ast U)
\]  

(45)

Lastly for any term \( \hat{s} \), consider Equation (46)’s sentence.

\[
\forall \, v \leq \hat{s} \, \Phi(v)
\]  

(46)

In this context for any term \( t \) (again required to be free in \( \Phi \)), the eighth elimination rule from Section 2 of [55] indicated that Equation (47) is allowed to be descendant of Equation (46)’s sentence in a semantic tableaux proof.

\[
t \leq \hat{s} \, \Rightarrow \, \Phi(t)
\]  

(47)

Since Equation (47)’s rule for eliminating universal quantifiers can apply to any term \( t \) (free in \( \Phi \)), it is applicable to the case where \( t \) is a term of the form “\( U \)” where \( U \) is a new constant created by Equation (43)’s elimination rule. In this special case, Equation (47)’s elimination rule can be rewritten as:

\[
U \leq \hat{s} \, \Rightarrow \, \Phi(U)
\]  

(48)

In essence, one may apply to Equation (41) \( n \) iterations of the elimination rules from Equations (43), (45) and (48) to construct a sequence of constants \( U_0, U_1, U_2, \ldots U_n \) such that \( U_0 = 2 \) and \( U_{i+1} = U_i \ast U_i \). (The footnote \(^8\) summarizes the structure of the \( i \)-th round of these \( n \) iterations.) In a formal sense, these \( n \) iterations may thus be simulated by a fragment of a semantic tableaux proof tree, denoted as \( F \), where all the branches of \( F \) are closed except for one branch, called the pivotal branch, which contains the parameter symbols of \( U_0, U_1, U_2, \ldots U_n \) together with a collection of sentences, appearing in linear order, asserting that \( U_0 = 2 \) and \( U_{i+1} = U_i \ast U_i \).

Hence this “pivotal branch” of \( F \) will imply that \( U_n = 2^{2n} \). Moreover since the fragment \( F \) will have only \( O(n) \) nodes, it will establish the

\(^8\) The \( i \)-th iteration of this process will have \( z, q \) and \( r \) from Equation (41) be replaced by respectively \( U_i \ast U_i, U_i \) and \( U_{i+1} \) via the elimination rules from respectively Equations (45), (48) and (43) to produce an essentially thrice-revised form of Equation (41) which implies that \( U_{i+1} = U_i \ast U_i \).
existence of a number $U_n = 2^{2^n}$, whose binary encoding has a $2^n$ length that is much larger than $F$’s length.

The above invariant is essentially all that we need to generalize [55]’s semantic tableaux version of the Second Incompleteness Theorem so that it also applies to Ax-4. (We obviously have omitted many details here. However, they are relatively straightforward extensions of [55]’s methodologies because the super-exponential growth property, established by the prior paragraphs, opens an avenue for introducing [55]’s proof techniques, whose formal details are too lengthy to be fully duplicated during this abbreviated proof sketch.) □

The remainder of this appendix will sketch how Lemma 5’s variant of the semantic tableaux version of the Second Incompleteness Theorem can be extended from the axiom system Ax-4 to Ax-3 (itself). Before doing so, we wish to introduce one further preliminary lemma.

**Lemma 6** The axiom system Ax-4 satisfies the same anti-threshold property for Herbrandized deduction as did Ax-3 (in Theorem 4).

**Proof Sketch** Every aspect of the proof of Theorem 4 (for Ax-3) does generalize also for Ax-4’s paradigm. This is because Theorem 4’s proof generalizes for any extension of Ax-3 that consists of a finite number of additional logically valid $\Pi^R_1$ sentences. Thus, if $\alpha$ denotes any such an extension of Ax-3 and if $\alpha^*$ denotes the extension of $\alpha$ which contains one additional $\Pi^R_1$ sentence, asserting the consistency of $\alpha^*$ (analogous to Theorem 3’s Diagonal($\alpha, D$) sentence), then every aspect of our prior analysis of Ax-3 applies also to $\alpha^*$. Thus using the same reasoning as before, it follows that $\alpha$ (and hence also Ax-4) must be anti-thresholds relative to Herbrand deduction. □

The combination of Lemmas 5 and 6 already shows that semantic tableaux and Herbrand deduction have polar opposite threshold properties with respect to Ax-4. (This is because the latter satisfies the semantic tableaux version of the Second Incompleteness Theorem, but it does not also satisfy its Herbrandized variant.) The same pair of polar opposite qualities also applies to $I\Sigma_0$’s Ax-3 axiomization. However, the proof that Ax-3 satisfies the semantic tableaux version of the Second Incompleteness Theorem is substantially more complicated than Lemma 5’s analogous result for Ax-4.

The final goal in this appendix will thus be to explain how one may incrementally revise Lemma 5’s proof-analysis for Ax-4 so that a similar incompleteness property also applies to Ax-3. In our discussion, $\overline{\psi}(v)$ will denote the following $\Delta^R_0$ formula, which is free only in $v$:

$$\forall q \leq v \ [ \ q \cdot q \leq v \ \Rightarrow \ \exists r \leq v \ ( \ r = q \cdot q) \ ]$$

(49)
In this notation, the sentence $\Psi$ that constituted Ax-4’s one additional $\Pi_1^R$ axiom-sentence (defined in Equation (41) ) can be rewritten as:

$$\forall v \, \widetilde{\psi}(v)$$  \hspace{1cm} (50)

Note that Ax-3, unlike Ax-4, does not contain Equation (50)’s $\Pi_1^R$ sentence as an axiom of its formalism. However using an analog of “passive induction” from our paper [55], we will show Ax-3 contains a counterpart of Equation (50) within its inductive schema that has comparable properties.

In particular, let $\psi(z)$ denote the following $\Delta_0^R$ formula:

$$\{ \widetilde{\psi}(0) \land \forall y \leq z \, [ \widetilde{\psi}(y) \implies \widetilde{\psi}(y') ] \} \implies \forall y \leq z \, \widetilde{\psi}(y)$$ \hspace{1cm} (51)

Then Equation (52) represents an encoding of one of Ax-3’s induction axioms.

$$\forall z \, \widetilde{\psi}(z)$$ \hspace{1cm} (52)

In order to explain the significance of this added notation, let $\Psi$ again denote Equation (41)’s axiom sentence (which we saw was identical to Equation (50)’s sentence except that the latter uses a slightly different notation). Also, let $\Psi^*$ denote Equation (52)’s sentence. In this notation, $\Psi^*$ can be viewed as a cousin of $\Psi$ which has the property that although $\Psi$ is not an axiom of Ax-3, its cousin $\Psi^*$ is a formal axiom of Ax-3.

The latter property is important because we need a vehicle for formally translating proofs from the Ax-4 system to proofs in the Ax-3 system. Since $\Psi$ is the only axiom belonging to Ax-4 which is not also in Ax-3, Equation (51)’s formal counterpart of it, called $\Psi^*$, will provide a means for doing this translation. The implications of this translation mechanism is described by our next lemma.

**Lemma 7** Let $\Omega$ denote an arbitrary theorem that is proven from the axiom system Ax-4 under a semantic tableaux proof $T$ that consists of $n$ applications of the $\forall$ elimination to rule to Equation (50)’s axiom sentence of $\Psi$. For some term $t_i$, let us assume that the $i$–th such application of this rule replaces Equation (50) with the reduced sentence of

$$\widetilde{\psi}(t_i)$$ \hspace{1cm} (53)

Then for a constant $k$ whose value is entirely independent of $n$, one can construct a proof $T^*$ of the same theorem $\Omega$ from the axiom system Ax-3.
where the difference between the number of node-sentences appearing in the proofs of \( T \) and of \( T^* \) is bounded by the quantity of \( kn \).

**Proof:** The justification of Lemma 7 is fairly straightforward. On each occasion where \( T \)’s proof contains a node similar to Equation (53), the comparable structure in \( T^* \) will replace this sentence with a tree-fragment that consists of the following four components:

1. An initial node-sentence of the form (54) (which is justified because it is an instance of Equation (52)’s axiom sentence of \( \Psi^* \)).

\[
\psi(t_i) \tag{54}
\]

2. A branch separation will appear directly below Equation (54)’s node that consists of the two sibling sentences of (55) and (56). In light of Equation (51)’s definition of \( \psi \), this binary separation is justified by the semantic tableaux rule for eliminating the \( \Rightarrow \) symbol (which was formalized by Item 4 from the second paragraph of [55]’s Section 2).

\[
\neg \left\{ \psi(0) \land \forall y \leq t_i \left[ \psi(y) \implies \psi(y') \right] \right\} \tag{55}
\]

\[
\forall y \leq t_i \neg \psi(y) \tag{56}
\]

3. Since Ax-3 can prove Equation (41)’s sentence of \( \Psi \) as a theorem, it is trivial to establish that for some fixed constant \( k_1 \) which does not depend on \( i \), one may insert a closed semantic tableaux proof of length \( k_1 \) under Equation (55)’s sentence, which accomplishes the desired effect of showing that Equation (55) is inherently contradictory.

4. Using one more time [55]’s semantic tableaux rules for eliminating the bounded universal quantifiers and the \( \Rightarrow \) symbol, one may easily insert a branch below Equation (56) that ends with the pair of sibling sentences given in equations (57) and (58) below. Moreover, it is evident that Equation (57) is inherently contradictory. Thus for some fixed constant \( k_2 \) (which again does not depend on \( i \)), the net consequence of this step will be to consist of a sequence of \( k_2 \) node sentences that includes the sentences given in (57) and (58) and closes the proof-tree that descends from Equation (57) with the desired forced contradiction.

\[
\neg \left( t_i \leq t_i \right) \tag{57}
\]

\[
\neg \psi(t_i) \tag{58}
\]

Let \( k \) denote a constant that equals the quantity of \( k_1 + k_2 + 3 \). Then each iteration of the above 4-step procedure will replace Equation (53)’s sentence with a subtree fragment consisting of \( k \) new sentences. Note that the final sentence at the end of each such iteration (given in Equation (58))
contains the identical logical statement as was given in Equation (53)’s initial sentence. Since all the other branches of our new sub-structure are closed and since Equations (53) and Equation (58) are identical to each other, it follows that after we finish performing \( n \) iterations of the above process, introducing \( kn \) new sentences, our new revised semantic tableaux tree will prove the theorem \( \Omega \) from Ax-3 instead of Ax-4. \( \square \)

**Theorem 5** The axiom system Ax-3 (similar to Ax-4) satisfies the semantic tableaux version of the Second Incompleteness Theorem. (Thus using Definition 1’s notation, Ax-3 is also a “threshold for the Second Incompleteness Effect” under semantic tableaux deduction).

**Proof Sketch:** For the sake of brevity, we will only outline the intuition behind Theorem 5’s proof. It will be essentially a consequence of the combination of Lemmas 5 and 7.

In particular, the core idea behind Lemma 5’s analysis of Ax-4 was to employ a sequence of \( n \) iterated applications of Equation (41)’s axiom of \( \Psi \) so as to construct a formal sequence of constants \( U_0, U_1, U_2, \ldots, U_n \) such that \( U_0 = 2 \) and \( U_{i+1} = U_i \# U_i \). By Lemma 7, the precisely identical sequence of \( U_0, U_1, U_2, \ldots, U_n \) can be constructed from Ax-3 using only \( kn \) additional sentences, for some fixed constant \( k \) whose value is independent of \( n \).

Since \( U_n \) represents the quantity of \( 2^{2^n} \) whose binary encoding has a length of \( 2^n \), this binary length is clearly much larger than \( kn \) as \( n \) approaches infinity. As a result of this exponential difference in lengths, it is relatively routine to revise Lemma 5’s proof of the semantic tableaux version of the Second Incompleteness Theorem for Ax-4 so that it may also apply to Ax-3. (The remaining details are analogous to the constructions that we used in \([50,55]\) and are omitted for the sake of brevity.) \( \square \)

**Summing Up What Has Been Done in this Appendix:** We have outlined abbreviated proofs showing that the Ax-3 and Ax-4 encodings for \( \Sigma_0 \) satisfy the semantic tableaux versions of the Second Incompleteness Theorem. We visited this topic twice because the result is easier to establish for Ax-4, although it is stronger for Ax-3. (The latter is stronger because Ax-3 contains one less axiom sentence than Ax-4.) Both our results for Ax-3 and Ax-4 are interesting because these formalisms are anti-thresholds for the Herbrandized version of the Second Incompleteness while they are thresholds when the paradigm is changed to focus upon semantic tableaux style deduction.

We close this appendix by once again reminding the reader that a different type of axiomatic setting where semantic tableaux and Herbrandized proofs have a sharp difference in length was formalized by Kołodziejczyk in \([18,19]\). (In particular, Kołodziejczyk \([18,19]\) focused his discussion on Buss’s bounded arithmetic of \([5]\).)
References


[36] R. Solovay, Private telephone conversations between Robert Solovay and Dan Willard (during April of 1994) concerning Solovay’s generalization of one of Pudlák’s theorems [31], using also some of Nelson’s and Wilkie-Paris’s
methodologies [25,46]. Solovay’s unpublished theorem shows that essentially no axiom system that recognizes Successor\(x) = x + 1\) as a total function (and which treats multiplication and addition as 3-way relations) can prove a theorem affirming its own consistency under Hilbert deduction. The Appendix A of [49] offers a 4-page interpretation of the intuition behind Solovay’s unpublished idea.


