

The Semantic Tableaux Version of the Second Incompleteness Theorem Extends Almost to Robinson's Arithmetic Q

Dan E. Willard

Dept. of Computer Science, University at Albany, dew@cs.albany.edu. **

Abstract. We will generalize the Second Incompleteness Theorem *almost to the level* of Robinson's System Q. We will prove there exists a Π_1 sentence V , such that if α is *any* finite consistent extension of $Q+V$ then α will be unable to prove its Semantic Tableaux consistency.

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1 Introduction

As originally formulated by Gödel, the Second Incompleteness Theorem discussed the inability of any possible extension of Peano Arithmetic (PA) to verify its self-consistency when it examined a proof-encoding that used a Hilbert-style methodology [5] as the underlying deductive calculi. Many generalizations of the Second Incompleteness Theorem were developed subsequently. For instance, let us recall that Robinson's Arithmetic system Q differs from Peano Arithmetic by containing no Induction axioms [10, 16]. In 1985, Pudlák strengthened a theorem by Bezboruah and Shepherdson [3] to show that Q was subject to many limitations similar to Peano Arithmetic. In particular, Pudlák established that if α is *any extension* of Q then α must be unable to prove the non-existence of a proof of $0=1$ when the proof formalism uses *simultaneously* a Hilbert-style calculi for deduction [5] and α represents its set of proper axioms. Moreover, Robert Solovay (private communications, [18]) combined several formalisms of Pudlák, Nelson [11] and Wilkie-Paris [22] to observe that Pudlák's theorem generalized to systems weaker than Q in that they would recognize Successor (but not Addition and Multiplication) as total functions.

The main gap left by the prior literature was that it had not resolved whether or not the Second Incompleteness Theorem was also valid for weak systems with respect to cut-free forms of deductive calculi, such as Semantic Tableaux. In particular, let us say an axiom system α has an ability to verify its **tableaux-based self-consistency** iff α can formally prove the non-existence of a proof of $0=1$ in a deduction system that uses **simultaneously** α as the set of proper axioms and Fitting's precise rules for Semantic Tableaux [5] as the employed deductive calculi. The prior literature had not clarified fully to what extent the Second Incompleteness Theorem prohibits weak axiom systems α from proving this particular notion of their self-consistency.

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For example, consider an axiom system that uses a special atomic symbol $M(x, y, z)$ to represent that the multiplicative product of x times y equals z . Such a system would not necessarily recognize that Multiplication is a “**total function**” (i.e. it need not recognize that “ $\forall x \forall y \exists z : M(x, y, z)$ ”). We devised in [23, 24] a consistent axiom system of this type, called IS(A), that could

1. recognize its tableaux-based self-consistency (in the particular sense that was defined on the preceding page).
2. recognize Addition as a total function, and
3. prove all the Π_1 -like theorems of Peano Arithmetic in a slightly modified language that replaces the the Multiplication Function symbol with a $M(x, y, z)$ -atomic predicate symbol.

A very different type of partial caveat for the Semantic Tableaux version of the Second Incompleteness Theorem has been described by the literature [8, 9, 11, 14, 20, 22]. The strongest version of this caveat has been described by Pudlák, and we will therefore use [6, 14]’s notation. Say a formula $\mathcal{T}(v)$ is a **definable cut** for an an axiom system α **if α can prove both** : $\mathcal{T}(0)$ and $\forall v \{ \mathcal{T}(v) \supset \mathcal{T}(v+1) \}$. Then [8, 9, 11, 14, 20] have illustrated several examples of different formulae $\mathcal{T}(v)$, which are Definable Cuts for α , such that α can prove its *Semantic Tableaux* consistency local to these Cuts. Thus if the notation “ $\text{SemPrf}_\alpha(x, y)$ ” designates that y is a semantic tableaux proof of the theorem x and if $\lceil \Psi \rceil$ designates Ψ ’s Gödel number, the prior literature has shown how several quite different axiom systems α can prove the **cut-localized statement of their** semantic tableaux consistency indicated below:

$$\forall y \{ \mathcal{T}(y) \supset \neg \text{SemPrf}_\alpha(\lceil 0 = 1 \rceil, y) \} \quad (1)$$

In light of these examples about how the Semantic Tableaux version of the Second Incompleteness Theorem can be partially evaded by certain types of formalisms, it is noteworthy that Adamowicz [1] has proven that the very-frequently-studied [6, 22] axiom system $I\Sigma_0 + \Omega_2$ satisfies the Second Incompleteness property for semantic tableaux. Since $I\Sigma_0 + \Omega_2$ has an ability to prove the totality of functions growing substantially faster than Multiplication, Adamowicz’s theorem raises the following two questions:

1. Where between the axiom system IS(A)’s evasion of the Second Incompleteness Theorem and $I\Sigma_0 + \Omega_2$ ’s obeying of it, does the Second Incompleteness effect become valid for Semantic Tableaux deduction?
2. Does the Second Incompleteness Effect for Semantic Tableaux proofs become valid for Robinson’s System Q, or other axiom systems very near it?

We will offer a partial (albeit not complete) answer to these questions. We will prove there exists a Π_1 sentence, henceforth called V , such that if α is *any* consistent and finite extension of $Q + V$ then α will be unable to formally prove a theorem asserting its own tableaux-based self-consistency. Moreover, this result will generalize for axiom systems α with infinite cardinality (although we will have insufficient space to prove this generalization here.)

2 Notation and Intuition

For simplicity, our base language will be the language of Robinson's Q. Thus only Addition and Multiplication function symbols will appear in its terms t , and the existential and universal quantifiers in " $\exists x \leq t \phi(x)$ " and " $\forall x \leq t \phi(x)$ " will be called *bounded quantifiers*. A formula will be called Δ_0 if all its quantifiers are bounded. Similarly if $\Psi(x_1, x_2, \dots, x_k)$ is Δ_0 , then we will use the term Π_1 to describe the normalized sentence $\forall x_1, \forall x_2 \dots \forall x_k \Psi(x_1, x_2, \dots, x_k)$.

In our discussion, $\text{SemPrf}_\alpha(x, y)$ will denote a Δ_0 formula indicating that y is a semantic tableaux proof of the theorem x from the axiom system α . For simplicity, we shall use a definition of Semantic Tableaux proof similar to that in Chapters 3.1 and 6.1 in Fitting's textbook [5]. However with one minor caveat, our version of the Second Incompleteness Theorem will actually generalize to all the other major definitions of semantic tableaux. The caveat is that each different definition of semantic tableaux deduction will be associated with a different Π_1 sentence V such that $Q + V$ is a threshold for the Second Incompleteness Theorem. (This is essentially because the Π_1 sentence V , constructed in the next section, will contain a Δ_0 subformula "SemPrf" that is dependent on the particular definition of Semantic Tableaux proof used.)

Given a sequence of integers $i_1, i_2, i_3, \dots, i_m$, we will say a Δ_0 formula $\Phi(g, j, x)$ makes g an encoding of this sequence iff $\Phi(g, j, x)$ is satisfied only when x represents the j -th element in the sequence $i_1, i_2, i_3, \dots, i_m$. We will say g is a **linear compressed encoding** of this sequence if $\text{Log}(g)$ has a magnitude proportional to the size of $\sum_{j=1}^m \text{Log}(i_j + 2)$. Buss, Hájek, Paris, Pudlák and Wilkie [4, 6, 22] have all given examples of such Δ_0 encodings, and we will therefore not also do so here. The Linear Compressed Encodings are considered to be the most efficient possible method to do a formal Gödel encoding of a sentence or proof, and we will therefore study it in this paper.

Let the symbol \perp denote the Gödel number of the sentence $0 = 1$. The *weakest possible definition* of α 's Semantic Tableaux Consistency is:

$$\forall p \neg \text{SemPrf}_\alpha(\perp, p) \quad (2)$$

If $\text{Def}(\alpha)$ denotes the above definition of α 's tableaux consistency then there are also other available definitions, denoted as say $\text{Def}^*(\alpha)$, such that the two definitions are equivalent in adequately strong fragments of arithmetic, *but not in sufficiently weak fragments* (see for instance [24–26]). It is preferable to employ the *weakest available definition* when generalizing the Second Incompleteness Theorems. This is why our present article uses Equation (2)'s definition.

An alternate definition of a "semantic tableaux proof" appears below:

Definition 1. Let $\text{Log}(x)$ denote Base-2 Logarithm, with downwards rounding to the lowest integer. Let $\text{Log}(x, k)$ denote $\text{Log}(\text{Log}(\text{Log} \dots (\text{Log}(x))))$ — where there are k iterations of logarithm. For any fixed constant $K > 1$, the symbol $\text{SemPrf}_\alpha^K(x, y, z)$ will denote a Δ_0 formula indicating that $\text{SemPrf}_\alpha(x, y)$ is valid **and** that $y < \text{Log}(z, K)$.

Oddly, our final theorem will be couched **solely** in terms of Equation (2)'s more desirable $\text{SemPrf}(x, y)$ formalism, but **the intermediate steps** of our proof will be much simplified by using as well Definition 1's " $\text{SemPrf}_\alpha^K(x, y, z)$ " formalism to shorten their analysis.

In particular, let $D(\alpha)$ denote the following diagonalization sentence:

* There is no Semantic Tableaux proof of **this sentence**.

Also, let $D^K(\alpha)$ denote the following $\text{SemPrf}_\alpha^K(x, y, z)$ modification of this diagonalizing sentence:

** In a context where one employs the slightly modified " $\text{SemPrf}_\alpha^K(x, y, z)$ " proof-notation, there exists no code (y, z) that "proves" **this sentence**.

It is very easy to formally encode $D^K(\alpha)$ as a Π_1 sentence following the example of the prior literature on diagonalization. Thus, let $\text{Subst}(g, h)$ denote Gödel's classic Δ_0 substitution relation, defined below:

$\text{Subst}(g, h)$ = The integer g is an encoding of a formula, and h encodes a sentence identical to g , except that all free variables in g are replaced with a constant, whose value equals g .

Then $D^K(\alpha)$ can be defined as being the Π_1 sentence $\Gamma(\bar{n})$, where $\Gamma(g)$ denotes the formula (3) and \bar{n} denotes $\Gamma(g)$'s Gödel number.

$$\forall h \forall y \forall z \{ \text{Subst}(g, h) \supset \neg \text{SemPrf}_\alpha^K(h, y, z) \} \quad (3)$$

In essence, our version of a proof of the Second Incompleteness Theorem will be similar to the classic diagonalization proofs, except that many intermediate steps will use $D^K(\alpha)$ instead of $D(\alpha)$.

Theorem 1. *Let $\alpha \vdash_S \Lambda$ denote that there is a semantic tableaux proof of the theorem Λ from the axiom system α . Then the combination of $\alpha \vdash_S \Lambda$, $\alpha \vdash_S \Theta$, and $\alpha \vdash_S \Lambda \wedge \Theta \supset \Xi$ implies $\alpha \vdash_S \Xi$.*

Proof. Immediate from Gentzen's Cut Elimination Theorem [19, 21]. \square

Theorem 2. *Suppose α is a finite extension of Robinson's system Q that (for some constant K) proves the three theorems below. Then α is inconsistent.*

- A) $\forall p \neg \text{SemPrf}_\alpha(\perp, p)$
- B) $\{ \exists y \exists z \text{SemPrf}_\alpha^K(\lceil D^K(\alpha) \rceil, y, z) \} \supset \exists x \text{SemPrf}_\alpha(\perp, x)$
- C) $\forall g \forall h \forall h^* \{ \text{Subst}(g, h) \wedge \text{Subst}(g, h^*) \} \supset h = h^*$

Proof. Let D^* denote the Gödel-like "diagonalizing" sentence below:

$$D^* =_{\text{df}} \{ \forall y \forall z \neg \text{SemPrf}_\alpha^K(\lceil D^K(\alpha) \rceil, y, z) \} \quad (4)$$

We will apply Theorem 1 several times to help shorten the proof of Theorem 2. First, let us apply Theorem 1 with Λ and Θ denoting the sentences from (A)

and (B) in Theorem 2's hypothesis and with Ξ representing the sentence D^* (from Equation (4)). For these three particular values for Λ , Θ and Ξ , it is immediately apparent that $\alpha \vdash_S \Lambda \wedge \Theta \supset \Xi$. Hence, Theorem 1 implies

$$\alpha \vdash_S D^* \quad (5)$$

For any fixed K , $D^K(\alpha)$ and D^* are certainly equivalent sentences in sufficiently strong models of Arithmetic. However, we need more than this fact to duplicate Gödel's diagonalization proof in the present setting. We need that *the weak axiom system α is yet strong enough* to also recognize this equivalence!

To establish this fact, we begin by recalling that \bar{n} denotes (3)'s Gödel number and that $D^K(\alpha)$'s definition implies $\text{Subst}(\bar{n}, [D^K(\alpha)])$ is true. Since $\text{Subst}(\bar{n}, [D^K(\alpha)])$ is a valid Δ_0 sentence and since Robinson's System Q can prove all valid Δ_0 sentences, it follows that Q can prove this sentence. Thus since Theorem 2's hypothesis indicates that α is an extension of Q, we get:

$$\alpha \vdash_S \text{Subst}(\bar{n}, [D^K(\alpha)]) \quad (6)$$

We will now use Equation (6) to infer the validity of (7) below. In particular, we do so by applying Theorem 1 with Λ representing the sentence $\text{Subst}(\bar{n}, [D^K(\alpha)])$, with Θ representing the sentence (C) in Theorem 2's hypothesis, and with Ξ being the identity $D^K(\alpha) \equiv D^*$. For these three values for Λ , Θ and Ξ , we can infer that $\alpha \vdash_S \Lambda \wedge \Theta \supset \Xi$ (because $\Lambda \wedge \Theta$ enables α to prove that the only value for h satisfying the left side of Equation (3) is the quantity $[D^K(\alpha)]$). Hence, Theorem 1 implies:

$$\alpha \vdash_S D^K(\alpha) \equiv D^* \quad (7)$$

Our final application of Theorem 1 is quite easy. We set Λ and Θ to be the sentences D^* and $D^K(\alpha) \equiv D^*$ and use Equations (5) and (7) to infer

$$\alpha \vdash_S D^K(\alpha) \quad (8)$$

We will now finish Theorem 2's proof by following Gödel's paradigm about proving a sentence that states roughly "*There is no proof of me*". Thus, let p denote (8)'s proof of the theorem $D^K(\alpha)$, q be a second integer satisfying $\text{Log}(q, K) > p$, and r denote $D^K(\alpha)$'s Gödel number. Let us also recall that if \bar{n} denotes (3)'s Gödel number, then $D^K(\alpha)$ is the sentence:

$$\forall h \forall y \forall z \{ \text{Subst}(\bar{n}, h) \supset \neg \text{SemPrf}_\alpha^K(h, y, z) \} \quad (9)$$

We will follow Gödel's example by observing that (9) must be false because if we replace its three variables y , z , and h with the three constants p , q , and r then (9)'s formal statement is negated. Moreover, Robinson's System Q is known to have the capacity to formally disprove any Π_1 sentence that is invalid in the Standard Model. Hence, since α is an extension of Q, we get

$$\alpha \vdash_S \neg D^K(\alpha) \quad (10)$$

The combination of (8) and (10) shows that α is inconsistent. \square

3 Main Theorems

We will assume that $x - y = 0$ when $x < y$ (so that Subtraction can be viewed as a total function). Also, $\text{Log}(x, u)$ was defined by Definition 1.

Lemma 1. *There exists two Δ_0 formulae $S(x, y, z)$ and $P(x, u, z)$ such that:*

1. *The graphs of $S(x, y, z)$ and $P(x, u, z)$ represent the set of ordered triples satisfying respectively the Subtraction and Logarithm functions*
2. *A Π_1 sentence, henceforth denoted as V_1 , will indicate that the two Δ_0 formulae $S(x, y, z)$ and $P(x, k, z)$ represent total functions assigning Subtraction and Logarithm their usual properties.*

Proof Sketch. Item 1's claims about Subtraction are obviously true. The unabridged version of this paper (which the author can mail to any interested readers) showed how $P(x, k, z)$ can encode $\text{Log}(x, k) = z$ as a Δ_0 formula. (Its proof essentially uses Benett's dissertation [2] and the theory of LinH functions [6, 7, 27]) in a routine manner.) The further claims of Item 2 require no proof because they are an immediate consequence of Item 1 and of the fact that Subtraction and Logarithm are non-growth functions.

Lemma 2. *Using the predicates $S(x, y, z)$ and $P(x, u, z)$ (from Lemma 1), one can easily encode four Π_1 sentences, henceforth denoted as A_1, A_2, A_3 and A_4 , that indicate that the functions Subtraction and $\text{Log}(x, k)$ have the following four well-known properties:*

1. $\forall x \forall y \forall z \quad xy \leq z \supset \text{Log}(z, 1) \geq \text{Log}(x, 1) + \text{Log}(y, 1)$
2. $\forall x \forall y \quad x^2 \leq y \supset \text{Log}(y, 2) \geq \text{Log}(x, 2) + 1$
3. $\forall x \forall y \quad \text{Log}(x, y + 1) = \text{Log}(\text{Log}(x, y), 1)$
4. $\forall x \forall y \quad y \leq x \supset [y = x \vee y \leq x - 1]$

Proof. A trivial consequence of Lemma 1. □

Let $\text{FinAx}(\alpha)$ denote a Δ_0 formula, which will return the Boolean value of TRUE when the integer α is a Gödel number that represents a finite-length list of logical sentences. The formula $\text{FinAx}(\alpha)$ will serve as a mechanism for recognizing when α represents a finite list of axioms. It thus allows us to treat “ $\text{SemPrf}_\alpha(x, y)$ ” as a Δ_0 formula with three input variables x, y, α (rather than just x and y). Likewise, “ $\text{SemPrf}_\alpha^k(x, y, z)$ ” will now denote (in this section) a Δ_0 formula free in five variables x, y, z, α and k .

Lemma 3. *There exists a Δ_0 formula, henceforth denoted as $\text{Map}(\alpha, k, d)$, which has the property that the triple (α, k, d) satisfies this formula if and only if d equals the Gödel number of Section 2's diagonalization sentence $D^k(\alpha)$. (This implies that $\text{Paradox}(y, z, \alpha, k)$, defined below, also has a Δ_0 encoding.)*

$$\text{Paradox}(y, z, \alpha, k) \stackrel{\text{df}}{=} \exists d < z \quad \text{Map}(\alpha, k, d) \wedge \text{SemPrf}_\alpha^k(d, y, z) \quad (11)$$

Justification. We will omit giving Lemma 3's proof here because its underlying structure is similar to Section 4 of our earlier paper [26]. It thus relies on

[6, 7, 27]'s theory of LinH functions. It uses the fact that every LinH procedure can be transformed into an analogous Δ_0 formula, in a context where the tuples satisfying $\text{Paradox}(y, z, \alpha, k)$ and $\text{Map}(\alpha, k, d)$ can be identified by LinH decision procedures.

We are now ready to define the “V” used by our axiom system $Q+V$. It can be thought of as either one long Π_1 sentence, representing a conjunction of five Π_1 clauses $V_1 \wedge V_2 \wedge V_3 \wedge V_4 \wedge V_5$ or as a list of five different Π_1 axiom-sentences. Its sentence V_1 was defined by Lemma 1. In our discussion, $\text{FinAx5}(\alpha)$ will denote a Δ_0 formula indicating α is a Gödel number which encodes some finite list of axioms that includes all the axioms of $Q + V$. Also, $\text{FinAx4}(\alpha)$ will have an identical definition as $\text{FinAx5}(\alpha)$, except it will require α 's list of axioms include only $Q + V_1 + V_2 + V_3 + V_4$. Below are defined V_2, V_3, V_4 and V_5 .

$$V_2 =_{\text{df}} A_1 \wedge A_2 \wedge A_3 \wedge A_4 \text{ where Lemma 2 defines these } A_i \quad (12)$$

$$V_3 =_{\text{df}} \{ \forall g \forall h \forall h^* \{ \text{Subst}(g, h) \wedge \text{Subst}(g, h^*) \} \supset h = h^* \} \quad (13)$$

$$V_4 =_{\text{df}} \{ \forall \alpha \forall k \forall g \forall h \forall y \forall z [\mathcal{Y}(\alpha, k, g, h, y, z) \supset \quad (14)$$

$$\exists h^* \leq h \exists y^* \leq y \exists z^* \leq z \mathcal{Y}(\alpha, k, g, h^*, y^*, z^*)] \}$$

$$\text{WHERE } \mathcal{Y}(\alpha, k, g, h, y, z) =_{\text{df}} \{ \text{Subst}(g, h) \wedge \text{SemPrf}_\alpha^k(h, y, z) \}$$

Comment: Since V_4 is provable from Q , some readers may wonder why it is needed? The answer is a redundant axiom can super-exponentially shorten the length of some cut-free proofs (a fact that we will use when we prove Lemma 9).

$$V_5 =_{\text{df}} \{ \forall y \forall z \forall \alpha \forall k \{ [\text{FinAx4}(\alpha) \wedge k \geq \alpha \wedge \text{Paradox}(y, z, \alpha, k)] \supset \quad (15)$$

$$\exists x < z \text{SemPrf}_\alpha(\perp, x) \} \}$$

Below are listed our two main theorems about $Q+V$.

Theorem 3. *The axiom system $Q+V$ is consistent.*

Theorem 4. *Suppose α is a consistent axiom system that is a finite extension of $Q+V$. Then α cannot prove a theorem asserting its Semantic Tableau consistency (i.e. it cannot verify $\forall x \neg \text{SemPrf}_\alpha(\perp, x)$).*

Comment : Before presenting the proofs of Theorems 3 and 4, it is desirable to briefly reflect upon our overall goals and strategies. The chief goal is, of course, to establish the existence of some Π_1 sentence V where $Q+V$ satisfies simultaneously Theorem 3's consistency property and Theorem 4's Incompleteness property. It turns out that there are many different Π_1 sentences V

that possess both these properties. Some of these V will oddly cause Theorem 3's half of the proof to be shorter, while others will cause Theorem 4's portion of the proof to be shorter. After some experimentation with this subject, we discovered that the paper's overall presentation would be best abbreviated if we chose to work with a particular axiom-sentence V where Theorem 4's portion of the proof would become extremely short, but Theorem 3's part of the proof would become much longer. The reason we are informing our readers about this trade-off is that some readers might otherwise find Theorem 4's proof (below) to be so alarmingly abbreviated to make them initially troubled by it. The best way to soothe such concerns is to remind our readers that it is possible to find some sentences V , where $Q + V$'s Incompleteness property has an astonishingly short proof, provided Theorem 3's proof of $Q + V$'s consistency property then becomes correspondingly longer.

Proof of Theorem 4. Suppose for the sake of contradiction the theorem was false. Then there would exist some fixed integer constant, denoted as say $\bar{\alpha}$, such that $\text{FinAx5}(\bar{\alpha})$ is true and where $\bar{\alpha}$, viewed as an axiom system, is consistent and verifies its own Semantic-Tableaux-Consistency.

Hence, we may begin our contradiction proof by letting $\bar{\alpha}$ represent a consistent axiom system satisfying

$$\bar{\alpha} \vdash \forall x \neg \text{SemPrf}_{\bar{\alpha}}(\perp, x) \quad (16)$$

Our goal is to show how Equation (16) will lead to a contradiction.

Let us now introduce a second constant $\bar{k} = \bar{\alpha} + 1$. We claim that the ordered pair $(\bar{\alpha}, \bar{k})$ will satisfy the hypothesis of Theorem 2. To establish this fact, we must show that $\bar{\alpha}$ can prove the three theorems (A), (B) and (C) required by Theorem 2's hypothesis. Below is the justification for this claim:

1. Equation (16) shows that $\bar{\alpha}$ can prove Item (A) from Theorem 2's hypothesis.
2. We will next show that $\bar{\alpha}$ also proves Theorem 2's needed sentence (B). Let A and Θ denote the following two sentences:

$$\text{FinAx4}(\bar{\alpha}) \wedge \bar{k} \geq \bar{\alpha} \wedge \text{Map}(\bar{\alpha}, \bar{k}, [D^{\bar{k}}(\bar{\alpha})])$$

$$\forall y \forall z \{ [\text{FinAx4}(\bar{\alpha}) \wedge \bar{k} \geq \bar{\alpha} \wedge \text{Paradox}(y, z, \bar{\alpha}, \bar{k})] \supset \exists x < z \text{SemPrf}_{\bar{\alpha}}(\perp, x) \}$$

Both these sentences are provable from $\bar{\alpha}$. In particular, $\bar{\alpha} \vdash A$ holds because $\bar{\alpha}$ can prove every Δ_0 sentence that is valid in the Standard Model of the Natural Numbers, and $\bar{\alpha} \vdash \Theta$ holds because Θ 's formal statement is identical to $\bar{\alpha}$'s V_5 axiom except that V_5 's universally quantified variables α and k are replaced by the constants $\bar{\alpha}$ and \bar{k} . Moreover, if we let Ξ denote the sentence (B) from Theorem 2's hypothesis, it is straightforward to obtain $\bar{\alpha} \vdash A \wedge \Theta \supset \Xi$ from the underlying structure of these three sentences. Hence from Theorem 1, we immediately obtain that $\bar{\alpha}$ can prove (B) from Theorem 2's hypothesis.

3. Since Theorem 2's sentence (C) and $\bar{\alpha}$'s axiom V_3 are the same sentence, $\bar{\alpha}$ can trivially prove (C).

Hence since $\bar{\alpha}$ satisfies Theorem 2's three requirements, the theorem implies $\bar{\alpha}$ is inconsistent. This observation completes our proof-by-contradiction because its first paragraph assumed $\bar{\alpha}$ was consistent. \square

4 The Consistency of Q+V

The axiom system Q+V and the preceding Theorem 4 would obviously both be entirely useless if Q+V was inconsistent. Therefore to establish their significance, we must prove Q+V's consistency. Our proof of Theorem 3 will require introducing several preliminary lemmas.

Lemma 4. *The first four axioms of V are valid in the Standard Model of the Natural Numbers, and these axioms can be written as Π_1 sentences.*

Proof. Lemmas 1 and 2 indicate the axioms V_1 and V_2 are valid Π_1 sentences in the Standard Model of the Natural Numbers. It is trivial that V_3 is also a valid Π_1 sentence. Finally for any Δ_0 formula $\phi(x, y)$, the sentence " $\forall d \forall e [\phi(d, e) \supset \exists f \leq e \phi(d, f)]$ " is clearly a valid Π_1 sentence. This implies that V_4 is also a valid Π_1 sentence, since its Equation (14) has a form identical to the preceding sentence except that it replaces d with (α, k, g) , e with (h, y, z) and f with (h^*, y^*, z^*) . \square

The remainder of this section will finish the proof of Theorem 3 by showing that the axiom V_5 satisfies a logical validity property similar to V_1 through V_4 .

Lemma 5. *If the clause on the left side of the axiom V_5 's implication symbol is true then $y < \text{Log}(z, 2^{3,000})$.*

Proof. The formula on the left side of the axiom V_5 's \supset symbol is:

$$\text{FinAx4}(\alpha) \wedge k \geq \alpha \wedge \text{Paradox}(y, z, \alpha, k) \quad (17)$$

Clearly $k \geq 2^{3,000}$ because (17) indicates that $k \geq \alpha$ and α must require more than 3,000 bits to encode an axiom system that includes $Q + V_1 + V_2 + V_3 + V_4$. Moreover since (17) indicates that $\text{Paradox}(y, z, \alpha, k)$ is satisfied, Equation (11) and Definition 1 imply $y < \text{Log}(z, k)$. \square

Lemma 6. *Let $\phi(x_1, x_2 \dots x_m)$ denote a Δ_0 formula. For any m -tuple of constants $(\bar{c}_1, \bar{c}_2 \dots \bar{c}_m)$ with each $c_i \leq n$, assume that the semantic tableaux proof (or disproof) of $\phi(\bar{c}_1, \bar{c}_2 \dots \bar{c}_m)$ from the axiom system α requires a tree with no more than s nodes. Let the symbol " $Q_i(x_i \leq \bar{k}_i)$ " be an abbreviation for either a bounded existential or universal quantifier. (In other words, " $Q_i(x_i \leq \bar{k}_i)$ " is an abbreviation for either " $\exists x_i \leq \bar{k}_i$ " or for " $\forall x_i \leq \bar{k}_i$ ".) Let Ψ denote a canonical Δ_0 sentence of the following form:*

$$Q_1(x_1 \leq \bar{k}_1), Q_2(x_2 \leq \bar{k}_2) \dots Q_m(x_m \leq \bar{k}_m) \{ \phi(x_1, x_2 \dots x_m) \} \quad (18)$$

Assume α is an extension of $Q + V_2$ and $n \geq \text{MAX}(\bar{k}_1, \bar{k}_2, \dots \bar{k}_m)$. Then it follows that:

1. If the sentence Ψ (formally defined by Equation (18)) is TRUE then there will exist a semantic tableaux proof of it of approximate size $O(s \cdot n^m)$.
2. And if the sentence Ψ is FALSE then there will analogously exist a semantic tableaux disproof of it of approximate size $O(s \cdot n^m)$.

Proof Sketch. Since all the quantifiers in Equation (18) are bounded quantifiers, it is immediately apparent that a brute-force algorithm can span the cross-product space of approximate size $O(n^m)$ and determine via a trivial procedure whether or not the sentence (18) is true or false. The fundamental point is that whenever an axiom system α is rich enough to include all four clauses of Equation (12)'s V_2 axiom, a corresponding semantic tableaux proof from it can simulate the exhaustive search paradigm and produce a similar-sized proof or disproof of (18) having an approximate $O(s \cdot n^m)$ length. (There is insufficient page space to provide more details about Lemma 6's proof here, but the preceding proof-sketch should be adequate to explain the main intuition.)

Definition 2. Given an axiom system α , say q is a **closed subtree rooted in Ψ** iff q has a structure identical to a semantic tableaux proof, except that the root of q consists of " Ψ " (rather than " $\neg \Psi$ ").

Lemma 7. Let $\bar{\alpha}$, \bar{k} and \bar{g} denote three constants and consider the following sentence:

$$\forall h \forall y \forall z \{ \text{Subst}(\bar{g}, h) \supset \neg \text{SemPr}_{\bar{\alpha}}^{\bar{k}}(h, y, z) \} \quad (19)$$

Suppose p is a Semantic Tableaux proof from $\bar{\alpha}$ of the theorem (19). Let Υ denote the formula from Equation (14). Then for any triple $(\bar{h}, \bar{y}, \bar{z})$, it is possible to map p onto a "closed" tableaux subtree q where:

- a. The root of q is " $\exists h^* \leq \bar{h} \exists y^* \leq \bar{y} \exists z^* \leq \bar{z} \Upsilon(\bar{\alpha}, \bar{k}, \bar{g}, h^*, y^*, z^*)$ "
- b. There will be a natural correspondence between the paths in p and the corresponding paths in q . Under this correspondence, the resulting path in q will have a length exceeding the path in p by no more than some constant C whose value is independent of $\bar{\alpha}$, \bar{k} , \bar{g} , \bar{h} , \bar{y} and \bar{z} . (Different authors use slightly different definitions of a "semantic tableaux deduction", and these minor variations will produce slightly different values of Lemma 7's constant C . Under a typical definition, such as say that in Fitting's textbook [5], C can represent a constant roughly $\cong 12$.)

Proof. It is obvious that (19) is provable only if the root of q 's subtree is false (and refutable). Thus, one would intuitively expect to be able to map the proof p onto an analogous tree-proof q of roughly the same height. The formal nature of this transformation is trivial, and omitted. \square

Definition 3. Given a natural number N , its **canonical binary representation**, denoted formally as " \underbrace{N} ", will be a term of length $O(\log N)$ that defines the value N using only the constant symbols for the numbers 0, 1 and 2. In particular, let b_0, b_1, \dots, b_m denote a sequence of bits where $N = \sum_{i=0}^m b_i 2^i$ and $b_m = 1$. Then \underbrace{N} will be the term:

$$(b_0 + 2 \cdot (b_1 + 2 \cdot (b_2 + 2 \cdot (\dots (b_{m-1} + 2 \cdot b_m))))))$$

Lemma 8. Suppose n, z and $e > 2$ satisfy $\text{Log}(z, e) > n$. Let Ψ denote a theorem which states “ $\exists r \text{ Log}(r, \underbrace{e}_n) > \underbrace{n}_n$ ” (where “Log” is obviously encoded using Lemma 1’s notation). Then there exists a constant C (whose value is independent of n, z and e) such that a semantic tableaux proof t of the theorem Ψ from the axiom system $Q + V$ can have its proof-length bounded by $O\{ [\text{LogLog}(z)]^C \}$.

Proof. Fitting’s definition of semantic tableaux [5] indicates that a proof of the theorem “ $\exists r \text{ Log}(r, \underbrace{e}_n) > \underbrace{n}_n$ ” will store the negation of this sentence in its root. Thus, the root will be essentially:

$$\forall r \text{ Log}(r, \underbrace{e}_n) \leq \underbrace{n}_n \quad (20)$$

The remainder of t ’s proof will consist of two fragments, which we shall denote as x_1 and x_0 . The substring x_1 will be the topmost section of the proof-tree t . It will use the fact that $Q + V$ recognizes multiplication as a total function to create a series of newly-created constant symbols $u_0, u_1, u_2, \dots, u_n$, satisfying $u_0 = 2$, $u_{i+1} = (u_i)^2$ and having its last term u_n satisfy $z < u_n \leq z^2$.

The second portion of the proof tree t will be called x_0 . It will use the last paragraph’s $u_n > z$ inequality to formally contradict the root’s sentence (stated in (20)). It is fairly straightforward to use the four clauses of the axiom V_2 to construct a formal semantic tableaux proof such that the proof substring x_1 will contain $O\{ \text{LogLog}(z) \}$ nodes, and x_0 will contain $O\{ [\text{LogLog}(z)]^C \}$ nodes (for some constant C). \square

Lemma 9. Suppose that the 4-tuple (y, z, α, k) satisfies the formula on the left side of the axiom V_5 ’s \supset symbol. This formula is duplicated below:

$$\text{FinAx4}(\alpha) \wedge k \geq \alpha \wedge \text{Paradox}(y, z, \alpha, k) \quad (21)$$

Then there will exist some constant C whose value is independent of y, z, α, k such that there exists a semantic tableaux proof x from α of the theorem $0=1$, where x ’s bit-length $\leq O\{ [\text{LogLog}(z)]^C \}$.

Proof. Although the statements of Lemmas 8 and 9 are very different, it turns out that Lemma 9’s proof tree x has some very similar components to Lemma 8’s tree t . In particular, x will be divided into five fragments, denoted as x_0, x_1, x_2, x_3, x_4 . The first two of these five components will be identical to the x_0 and x_1 fragments of Lemma 8’s proof tree t .

Some notation will clarify the differences between these two proof trees. If x_i denotes any one of x_0, x_1, x_2, x_3, x_4 , let us use the following terminology:

1. The fragment x_i , viewed as an integer encoding a string of bits, will be said to be **z-tiny** iff $\text{Log}(x_i) \leq O(\text{LogLogLog}(z))$ (there are *three* iterations of “Log” attached here to z).

2. The bit-string x_i will be said to be **z-adequately small** when x_i satisfies the inequality $\text{Log}(x_i) \leq O \{ [\text{LogLog}(z)]^C \}$.

Our proof-tree x will be essentially the concatenation of two “z-adequately small” parts, x_0 and x_1 , with three “z-tiny” subtrees, x_2 , x_3 and x_4 . As we already noted, x_0 and x_1 will be identical to their counterparts from Lemma 8’s tree t . (Thus, it will turn out that the sole differences between the proof-trees, t and x , will be three very minuscule-sized “z-tiny” fragments.)

We will now formally describe the proof-tree x . Since x is a proof of the theorem $0=1$, the root of its proof-tree will obviously be “ $0 \neq 1$ ” . Immediately below this root will be the fragment x_1 from Lemma 8’s proof. It will thus consist of an iterated series of newly-created constant symbols $u_0, u_1, u_2, \dots, u_n$, satisfying $u_0 = 2$, $u_{i+1} = (u_i)^2$ and having its last term u_n satisfy

$$z < u_n \leq z^2 \tag{22}$$

At the bottom of the proof fragment x_1 will appear two further sentences. The first will be Equation (14)’s axiom V_4 . (It is permissible to include this axiom in the proof x because Equation (21)’s $\text{FinAx4}(\alpha)$ clause indicates that the axiom system α includes this axiom.) The last sentence at the bottom of the segment x_1 is formally defined by Equation (23) at the end of this paragraph. This last sentence will be identical to the axiom V_4 , except that V_4 ’s six universally quantified variables will now be replaced by six terms, denoted as $\underbrace{\alpha}$, \underbrace{k} , \underbrace{g} , \underbrace{h} , \underbrace{y} and u_n . The following rules will define these terms:

1. Let us recall that Lemma 9’s hypothesis indicated that (y, z, α, k) would satisfy Equation (21). The values of $\underbrace{\alpha}$, \underbrace{k} and \underbrace{y} will represent the corresponding quantities in the tuple (y, z, α, k) . Formally, these three terms will be encoded using the “Canonical Binary” form of Definition 3. (The last term u_n will obviously satisfy Equation (22).)
2. Let us recall that Equation (21)’s $\text{Paradox}(y, z, \alpha, k)$ formula indicates that y is a proof of the theorem $D^k(\alpha)$. Let h denote $D^k(\alpha)$ ’s Gödel number. We saw previously in Equation (3) of Section 2 how h was constructed. (It basically was constructed using methods similar to Gödel’s construction of a sentence that states “*There is no proof of me*”. In particular, h was constructed by taking a “masking formula” g , that was free in one variable, and letting h denote the unique integer that satisfies the Gödel-like formula of $\text{Subst}(g, h)$.) Our formal definition of the terms \underbrace{g} and \underbrace{h} is that they will be canonical binary terms that represent the particular integer values associated with these two numbers g and h .

Thus in a context Items (1) and (2) define the terms $\underbrace{\alpha}$, \underbrace{k} , \underbrace{g} , \underbrace{h} , \underbrace{y} and u_n , the node immediately below V_4 in our proof tree will be identical to V_4 , except that V_4 ’s six universally quantified variables of α, k, g, h, y and z will be replaced by these corresponding terms. This sentence is shown below.

$$\mathcal{T}(\underbrace{\alpha}, \underbrace{k}, \underbrace{g}, \underbrace{h}, \underbrace{y}, u_n) \supset \exists h^* \leq \underbrace{h} \exists y^* \leq \underbrace{y} \exists z^* \leq u_n \mathcal{T}(\underbrace{\alpha}, \underbrace{k}, \underbrace{g}, h^*, y^*, z^*) \} \quad (23)$$

First Branch-Split in the Proof x Immediately below the node storing the sentence (23) will occur the first branch-split in x 's semantic tableaux proof. This split will be generated by the \supset Elimination Rule. The two children of (23)'s sentence are therefore given by equations (24) and (25) below:

$$\exists h^* \leq \underbrace{h} \exists y^* \leq \underbrace{y} \exists z^* \leq u_n \mathcal{T}(\underbrace{\alpha}, \underbrace{k}, \underbrace{g}, h^*, y^*, z^*) \} \quad (24)$$

$$\neg \mathcal{T}(\underbrace{\alpha}, \underbrace{k}, \underbrace{g}, \underbrace{h}, \underbrace{y}, u_n) \quad (25)$$

Construction of the Substring x_2 and the Proof of its z-Tiny Size: The substring x_2 will represent a closed subtree of z-tiny size that is rooted in Equation (24)'s sentence. The reason it is possible to build x_2 is that the hypothesis of Lemma 9 indicated that $\text{Paradox}(y, z, \alpha, k)$ was satisfied. From Lemma 7A, this implies x_2 exists. Moreover, its size must be z-tiny (because Lemma 7B implies x_2 has the same magnitude as y and Lemma 5 indicated y was actually much smaller than z-tiny).

The Remaining Fragments of the Proof-Tree x and Their Small sizes: To complete Lemma 9's proof, we must show how the subtree descending from (25) is also closed and adequately small. This sentence is the statement " $\neg \mathcal{T}(\underbrace{\alpha}, \underbrace{k}, \underbrace{g}, \underbrace{h}, \underbrace{y}, u_n)$ " where \mathcal{T} was defined by (14). After applying the Semantic-Tableaux \neg Elimination Rule to the preceding quoted sentence, our remaining part of the proof-tree is rooted in the sentence

$$\neg \text{Subst}(\underbrace{g}, \underbrace{h}) \vee \neg \text{SemPrf}_{\underbrace{\alpha}}^k(\underbrace{h}, \underbrace{y}, u_n) \quad (26)$$

Applying the \vee -Elimination Rule to (26), we get a branch-split generating the following two sibling nodes:

$$\neg \text{Subst}(\underbrace{g}, \underbrace{h}) \quad (27)$$

$$\neg \text{SemPrf}_{\underbrace{\alpha}}^k(\underbrace{h}, \underbrace{y}, u_n) \quad (28)$$

Moreover from Definition 1, it is apparent we can make another branch split below (28) that will generate the sibling nodes given in (29) and (30).

$$\neg \text{SemPrf}_{\underbrace{\alpha}}(\underbrace{h}, \underbrace{y}) \quad (29)$$

$$\neg \underbrace{y} < \text{Log}(u_n, \underbrace{k}) \tag{30}$$

Thus to show that the subtree descending from (25) is closed and also sufficiently small, we must analyze the three subtrees that descend from the corresponding sentences (27), (29) and (30). These three subtrees will be called x_3 , x_4 and x_0 , and their size analysis is given below:

1. The quantities g and h from (27)'s formula “ $\text{Subst}(\underbrace{g}, \underbrace{h})$ ” must be integers less than y (simply because y is a proof of the theorem whose Gödel number = h). Since, Subst is a Δ_0 formula, it then follows from Lemma 6 that the subtree x_3 descending from (27) has a size approximately equal to y 's magnitude. The final point is that Lemma 5 indicated that y was actually much smaller than z -tiny. Hence, the subtree x_3 must certainly have a z -tiny magnitude (since its length is governed by y 's size).
2. The proof that the subtree x_4 descending from (29) is z -tiny is almost identical to Item 1's analysis of x_3 essentially because $\text{SemPrf}_\alpha(\underbrace{h}, \underbrace{y})$ is also a Δ_0 formula. Thus once again, Lemma 5 allows us to assume that y (and therefore again also h) are sharply smaller than z -tiny. Also, an obvious analog of Lemma 5 is similarly applicable to α . Hence since all three of $\text{SemPrf}_\alpha(\underbrace{h}, \underbrace{y})$'s terms are sufficiently small, we can again apply Lemma 6 to conclude that the subtree x_4 is z -tiny.
3. The proof that the subtree x_0 descending from (30) is z -adequately small is essentially an immediate consequence of Lemma 8. In particular, the tree t , constructed in Lemma 8's proof, contained a subtree, which was also called x_0 , whose structure is identical to the object descending from (30)'s sentence. Thus by our preceding discussion, x_0 is z -adequately small.

The above observations have completed our proof of Lemma 9 because they have shown that each of the five subtrees x_0, x_1, x_2, x_3, x_4 are sufficiently small to satisfy Lemma 9 □

Lemma 10. *The axiom V_5 is valid in the Standard Model of the Natural Numbers.*

Proof Sketch. Almost all the details needed to establish Lemma 10 already appeared in Lemma 9's proof. This is because the formal statement of Lemma 9 is almost identical to the formal statement of axiom V_5 . Thus Lemma 9's formal statement indicated that any tuple (y, z, α, k) satisfying the formula (21) can be mapped onto an element x , representing a semantic tableaux proof of $0=1$, whose bit-length is bounded by $O\{[\text{LogLog}(z)]^C\}$. The point is that V_5 's formal statement differs only by indicating that $x < z$.

The additional details to convert Lemma 9's $\text{Log}(x) < O\{[\text{LogLog}(z)]^C\}$ bound into a strict $x < z$ inequality are extremely routine, albeit a bit tedious. For the sake of brevity, these final details are omitted. □

Finishing the Proof of Theorem 3: The combination of Lemmas 4 and 10 implies Theorem 3's validity because they show all $Q + V$'s axioms are valid in the Standard Model (hence establishing its consistency). □

Recapitulating This Proof: It is now possible to offer a pleasantly short 2-sentence intuitive summary of Theorem 3's proof. The heart of the proof rested on showing the existence of a proof-tree x that was sufficiently small to assure the validity of the axiom V_5 's requirements for x . The intuitive reason such an x *must exist* is that its bit-length has the same order of magnitude as Lemma 8's proof-tree t , and Lemma 8 showed the latter was sufficiently small.

5 Final remarks

Section 1 explained the basic reason for our interest in this subject. It was because our prior papers [23, 24] established that significant exceptions to the semantic-tableaux version of the Second Incompleteness Theorem do occur when one transforms Multiplication from a total function into a 3-way relation. (Thus, our prior papers [23, 24] showed that some axiom systems can verify **all** Peano Arithmetic's Π_1 theorems while *simultaneously* proving their *Semantic Tableaux* consistency when Multiplication is changed from a total function into a 3-way relation.) Since Theorem 4 isolates a single Π_1 sentence **preventing the same** effect when Multiplication serves as a total function, the combined results of our several papers establish a fairly tight characterization of exactly when the Second Incompleteness Theorem **is and is not** valid for Semantic Tableaux.

It should be emphasized the particular Π_1 sentence V defined by equations (12) through (15) is not the main point, since Section 4 noted many other Π_1 sentences can also satisfy Theorems 3 and 4. Rather what is noteworthy is that this paper has established the first documented example of some Π_1 sentence V with these Second Incompleteness properties.

A slightly longer version of the present article has also been prepared, and it can be emailed to any interested readers. It includes fuller proofs for Lemmas 1 and 10 and a stronger version of Theorem 4 (which indicates that the Semantic Tableaux Incompleteness Effect *is also valid* for axiom systems α *with infinite cardinality*). Moreover, with a little additional work, our variant of the Second Incompleteness Theorem can also be generalized to other cut-free methods of deduction, such as Herbrand deduction, the cut-free version of the Sequent Calculus, and Resolution.

Finally, we wish to end this article on a mildly amusing note. Many readers will smile with amusement when they learn the true reason that the Semantic Tableaux version of the Second Incompleteness Theorem breaks down when Multiplication is changed from a total function into a 3-way relation. It is essentially that Lemma 8 and its short proof then become no longer valid. (Lemma 8 was crucial because Lemma 9's proof used the x_0 and x_1 substrings from Lemma 8's proof as an interim step. Without it, our proof of the Semantic Tableaux version of the Second Incompleteness Theorem *collapses entirely!*)

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