On the Partial Respects in which a Real Valued Arithmetic System Can Verify its Tableaux Consistency

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1 Introduction

Let $A(x, y, z)$ and $M(x, y, z)$ denote two 3-way predicates indicating $x + y = z$ and $x * y = z$. An axiom system α will be said to **recognize** successor, addition and multiplication as **Total Functions** iff it can prove (1), (2) and (3).

$$\forall x \, \exists z \;\; A(x, 1, z) \tag{1}$$

$$\forall x \, \forall y \, \exists z \;\; A(x, y, z) \tag{2}$$

$$\forall x \, \forall y \, \exists z \;\; M(x, y, z) \tag{3}$$

It is known that Equations (1) – (3) are related to both generalizations of Gödel's Second Incompleteness Theorem and to its boundary-case exceptions. For instance, Equation (1) will enable the Second Incompleteness Theorem to apply to Hilbert deduction. Also, [36, 38] showed that the semantic tableaux version of the Second Incompleteness Theorem generalizes for essentially all axiom systems that can prove the validity of Equations (1) – (3) for integer arithmetic. On the other hand, [35, 38, 39, 41] showed exceptions to the semantic tableaux version of the Second Incompleteness Theorem do exist when an axiom system fails to support Equation (3).

The preceding research naturally raises the question whether or not an analogous phenomenon holds when one changes the venue of application from integer arithmetic to the addition and multiplication operations of a computer's floating point arithmetic set. Throughout this paper, we will use the term *simulated real-arithmetic* to refer to an instruction set that is slightly more general and powerful than the common floating point instructions on a digital computer's hardware. We will prove that simulated real arithmetic is *quite unlike* integer arithmetic — insofar as an axiom system can simultaneously recognize its semantic tableaux consistency and the validity of Equations (1) – (3) *for simulated real arithmetic.*

This result is significant because a computer's floating point instruction set has essentially as many practical applications as an integer arithmetic. Moreover, Section 5

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will formalize another very unusual aspect of simulated arithmetic. It will be that our partial exceptions to the Second Incompleteness Theorem actually house a novel type of limited Gentzen-style deductive cut rule for simulated real arithmetic, *whose analog for integer-based arithmetics is infeasible.* Thus, our main contribution will be the demonstration that simulated real-valued arithmetic *in several respects* supports more *robust forms* of tableaux-like exceptions to Gödel's Second Incompleteness Theorem than an integer arithmetic can feasibly do.

2 Literature Survey

Gödel's 1931 paper on incompleteness [10] contained two major results, called the "First" and "Second" Incompleteness Theorems. The former theorem (after Rosser [23] strengthened it) established that it was impossible to construct a consistent r.e. axiomatic extension of Peano Arithmetic that could prove or disprove every logical sentence.

Gödel's "Second Incompleteness Theorem" showed that neither Peano Arithmetic nor any extension of it can confirm its own consistency. During the last 75 years, there has been a substantial effort to determine under exactly what circumstances, Gödel's Second Incompleteness Theorem is applicable to axiom systems weaker than Peano Arithmetic. In our summary of this topic, it is it is useful to classify an *"integer" arithmetic* axiom system α as being of:

a. **Type-M** iff α can formally prove the assertions in Equations (1) – (3), indicating that "integer" multiplication, addition and successor are "total" functions.
b. **Type-A** iff α can prove that integer addition and successor satisfy Equations (1) and (2), but α cannot formally prove integer multiplication satisfies (3).
c. **Type-S** iff α can prove successor is a total function (as specified by Equation (1)), but it is uncertain about whether integer arithmetic satisfies (2) and (3).
d. **Type-NS** iff α is unable to prove any of Equations (1), (2) or (3).

The research into generalizations of the Incompleteness Theorem for weak axiom systems began with Tarski-Mostowski-Robinson [30]'s observation that a Type-M axiom system, which they called Q, had the property that Q (but none of its proper subsets) satisfied a condition called "essential undecidability". Also, Bezboruah-Shepherdson [6] proved a version of the Second Incompleteness Theorem for Q which stated that Q was unable to prove some types of particularized theorems affirming its own Hilbert consistency. The subsequent research can be roughly summarized as follows:

I. Pudlák (1985) made the seminal observation [20] that the Bezboruah-Shepherdson [6] result could be generalized for all extensions of Q, all encodings of its Hilbert-proof predicate, and also for all its local extensions in the range of a specified "Definable Cut". Wilkie-Paris [34] further noted that the axiom system $I\Sigma_0 + Exp$ is unable to prove the Hilbert consistency of Q. Nelson [17] observed that many of the properties of the Tarski-Mostowski-Robinson [30] axiom system Q can be extended to also hold for Type-S axiom systems. In 1994, Solovay [26] had combined these three formalisms to obtain the following result:

Theorem 1. (Solovay's modification [26] of Pudlák's Theorem 2.3 from [20] using the additional mathematical methodologies of Nelson and Wilkie-Paris [17, 34].) *No consistent Type-S axiom system, formalizing integer addition and multiplication as 3-way relations A(x,y,z) and M(x,y,z), can prove the non-existence of a Hilbert-proof of* $0 = 1$ *from itself.*

II. An open question that Paris-Wilkie [19] posed in 1981 was whether an analog of Theorem 1 held for cut-free methods of deduction, such as semantic tableaux. Adamowicz-Zbierski [1, 4] answered this question in the positive for the axiom system $\mathrm{I}\Sigma_0 + \Omega_1$, and Willard extended their result for $\mathrm{I}\Sigma_0$ and yet weaker derivatives of this system. The main result of [36, 38] was to construct a Π_1 sentence V such that every consistent Type-M system $\alpha \supset V$ is necessarily unable to prove a theorem affirming its semantic tableaux consistency. (Here and elsewhere in this paper, our definition of semantic tableaux is similar to that in Fitting's textbook [9].

A more detailed literature review about the properties of the Second Incompleteness Theorem is provided in [41]. Since the sundry generalizations of the Second Incompleteness Theorem are quite powerful, we have *cautiously* called our partial evasions of the Second Incompleteness Effect *"boundary-case exceptions"*. These exceptions have consisted mostly of Type-NS systems [37] that can verify their Hilbert consistency and Type-A systems [37, 39] that can verify their tableaux-oriented consistencies. These results are near-maximal because the Type-NS evasions closely interface against Item I's generalization of the Second Incompleteness Theorem for Type-S systems, and similarly Item II's variant of the Second Incompleteness Theorem for Type-M systems complements [35, 37, 39]'s Type-A evasions.

Our new results in the current article will differ from the prior work by *changing the venue of application* from integer arithmetic to a computer's simulated real-valued arithmetic instruction set, so that our tableaux-style partial exceptions to the Second Incompleteness Theorem shall be able to recognize *both* addition and multiplication as total functions for simulated real arithmetic. Aside from directly eschewing Item II's integer-based form of the Second Incompleteness Theorem (given above), this approach will also help clarify the nature of numerical analysis's logical foundations.

Let us now assume α is an axiom system and D a deduction method. The pair $(\, \alpha\, ,\, D\,)$ will be called an **Introspectively Unified Logic** iff

A) one of α's formal theorems will state that the deduction method $D,$ applied to the axiom system $\alpha,$ will produce a consistent set of theorems,

B) and the axiom system α is in fact consistent.

Also, an axiom system α will be called **Self-Justifying** iff there exists some frequently employed deduction method $D,$ such as perhaps the tableaux, Hilbert, Herbrand or sequent-calculus formalisms, where $(\, \alpha\, ,\, D\,)$ is an introspectively unified logic.

Kleene's Fixed Point Theorem implies that every r.e. axiom system α can be easily extended into a broader system α^* which satisfies Part-A of the definition for introspective unification. For a fixed deduction method $D,$ Kleene essentially proposed [14] to let α^* contain all α's axioms plus one additional axiom sentence stating :

$+$ There is no proof (using deduction method D) of a prototypical absurdity-sentence (such as for example "0=1") from the union of the axiom system α with $this$ sentence "+"(looking at itself).

Kleene explained how to apply the Fixed Point Theorem to encode a self-referencing statement, similar to the axiom (above). However, he pointed out that the catch is that α^* may be inconsistent even while its added axiom formally asserts α^* 's consistency.

For this reason, Kleene, Rogers and Jeroslow [13, 14, 22] each warned their readers that most axiom systems, similar to α^*, were useless on account of their inconsistency, *although they were technically well-defined.* In the notation used in this section, the difficulty is that (α^*, D) may satisfy Part-A but not the equally important Part-B of the definition of "introspective unification".

This problem arises in settings considerably more general than Gödel's original paradigm, where α was an extension of Peano Arithmetic. Thus, [1–6, 8, 12, 18–21, 24–26, 29–31, 34, 36, 38, 40, 42] discuss a variety of generalizations of the incompleteness theorem where a similar paradigm applies to weak axiom systems.

Our interest in this topic began [35, 37, 39] with the observation that there are certain well-defined settings where it is feasible to construct introspectively unified logics (α , D) using the Kleene-like "I am consistent" axioms, analogous to the sentence $+$. In particular, boundary-case exceptions to the Second Incompleteness Theorem do exist when either $1)$ α is a Type-NS axiom system and D represents Hilbert deduction, or when $2)$ α is a Type-A integer arithmetic and D represents some cut-free deductive method, such as a tableaux formalism. We again remind the reader that these two results are near maximal on account of [20, 26, 36, 38]'s Type-S and Type-M generalizations of the Second Incompleteness Effect, summarized earlier by Items I and II of this section.

Prior to our research, the proof-theoretic literature has sought to partially evade the Second Incompleteness Theorem largely by studying what perhaps can be called the localized $(\alpha, D, \varphi)-$consistency statements. In particular, let $\lceil \Psi \rceil$ denote a formula Ψ 's Gödel number, and $\text{Prf}_\alpha^D (t, p)$ denote p is a proof of the theorem t from the axiom system α using the deduction method D . Let us say α can recognize its **localized** $(\alpha, D, \varphi)-$**consistency property** if α can prove the theorem:

$$\forall\, p \quad \{ \;\; \varphi(p) \;\; \rightarrow \;\; \neg\, \text{Prf}_\alpha^D \; (\; \lceil 0 = 1 \rceil \; , \; p \;) \;\; \} \tag{4}$$

In order to analyze Equation (4)'s meaning, let **Sequence(**φ) denote the formula:

$$\varphi(0) \quad \text{AND} \quad \forall\, x \quad \varphi(x) \; \rightarrow \; \varphi(x+1) \tag{5}$$

Let **Induction(**φ) denote (6)'s statement about the validity of the principle of induction:

$$\text{Sequence}(\varphi) \quad \rightarrow \quad \forall x \; \varphi(x) \tag{6}$$

It is clear that any conventional axiom system that can simultaneously prove Equations (4) – (6) will be able to combine these results to infer its own global consistency (and thereby evade the Second Incompleteness Theorem). To avoid these effect, [11, 12, 15–21, 25–27, 31–34] have considered localized $(\alpha, D, \varphi)-$consistency statements that come tantalizingly close to such an evasion via their axiom systems retaining an ability to prove Equations (4) and (5) *but not the also-needed* Equation (6). The strongest results about this subject matter were derived by Pudlák [20], who showed that if α represents essentially any axiom system of finite cardinality and D denotes either the Herbrand or tableaux deduction method, then there exists formulae $\varphi(x)$ where α has the tantalizing property that it can prove the first two of the preceding three wffs.

It is difficult to make detailed comparisons between our self-justifying formalisms, which employ the Statement $+$'s "I am consistent axiom", with the literature about localized consistency because each approach has its own separate objectives, advantages and difficulties when it seeks to evade the Second Incompleteness Theorem. Thus, self-justifying systems are unable to recognize *integer* multiplication as a total function, whereas the comparable sacrifice in the literature concerning localized consistency is that sentences, similar to Equation (4), have their meaning diluted when α is unable to prove Induction(φ). In both cases, it is of theoretic interest to categorize the maximal types of partial evasions of the Second Incompleteness Theorem that are feasible.

3 General Self-Justification Framework

Define a mapping $F(a_1, a_2...a_j)$ to be a **Non-Growth** function iff it satisfies the invariant $F(a_1, a_2, ...a_j) \leq Maximum(a_1, a_2, ...a_j)$. Six examples of non-growth functions are *Integer Subtraction* (where $x - y$ is defined to equal zero when $x \leq y$), *Integer Division* (where $x \div y$ is defined to equal x when $y = 0$, and it equals $\lfloor x/y \rfloor$ otherwise), $Maximum(x, y)$, $Logarithm(x)$, $Root(x, y) = \lceil x^{1/y} \rceil$ and $Count(x, j)$ designating the number of "1" bits among x's rightmost j bits. These function are called the **Grounding Functions**.

The term **U-Grounding Function** will refer to a set of eight operations, which includes the six non-growth "Grounding" functions plus the *growth operations* of addition and *Double*$(x) = x + x$. For simplicity, we will use a U-Grounding language. Its notation is technically unnecessary — because a system that uses Equation (7) (which specifies addition is a total function) along with our first six *non-growth* Grounding operations would have properties similar to the U-Grounding language.

$$\forall x \, \forall y \, \exists z \quad x = z - y \tag{7}$$

However, the U-Grounded notation makes it easier to present our results.

Our formal analogs for Logic's Π_n and Σ_m sentences in the U-Grounding language will be called Π_n^* and Σ_m^*. Here, a *term* t is defined to be a constant, variable or a U-Grounding function symbol (whose input arguments are recursively defined terms). Also, the quantifiers in the wffs $\forall \, v \leq t \;\; \Psi(v)$ and $\exists \, v \leq t \;\; \Psi(v)$ are called *bounded integer quantifiers*. Any formula in the U-Grounding language, all of whose quantifiers are bounded, will be called Δ_0^*. Following conventional notation, every Δ_0^* formula will be considered to satisfy the " Π_0^* " and " Σ_0^* " conditions. For $n \geq 1$, a formula Υ shall be called Π_n^* iff it is written in the form $\forall v_1 \, \forall v_2 \, ... \, \forall v_k \;\; \Phi$, where Φ is Σ_{n-1}^*. Likewise, Υ is called Σ_n^* iff it is written in the form $\exists v_1 \, \exists v_2 \, ... \, \exists v_k \;\; \Phi$, where Φ is Π_{n-1}^*. Henceforth, we will also use the following definitions:

1. A **Level(n) Definition** of an axiom system α's tableaux consistency is the declaration that there exists no Π_n^* sentence Υ supporting simultaneous semantic tableaux proofs from α of both Υ and its negation.
2. A **Level**($0-$) **Definition** of a system α's tableaux consistency is the statement that there exists no proof of 0=1 from α.

We will also sometimes use a notation referring to Π_n^- and Σ_n^- sentences. These sentences will have the same definitions as Π_n^* and Σ_n^* except that they will use a language employing the Grounding (instead of U-Grounding) functions.

In essence, both notations are useful. The U-Grounding based notation is preferable when an axiom system uses Equation (7)'s axiom, declaring addition is a total function. It then leads to more succinct proofs. On the other hand, the slightly more cumbersome Π_n^- and Σ_n^- notation is essential for axiom systems that do not recognize addition as a total function. In essence, we will employ the U-Grounding notation in Sections 3 and 5 and the Grounding language notation in Section 4.

Theorem 2. . (A summary of the central results published in [35, 37, 39]) *Let A denote an arbitrary consistent axiom system employing the U-Grounding language. Then it is possible to construct two further consistent axiom systems, called IS(A) and IS-1(A) [35, 37, 39], having the following properties.*

1. *Both IS(A) and IS-1(A) will retain a capacity to prove all the Π_1^* theorems of A ,*
2. *Both IS(A) and IS-1(A) will recognize addition as a total function,*
3. *IS(A) will retain an ability to recognize its own Level(0-) consistency, and IS-1(A) will have a stronger ability to also recognize its Level(1) consistency.*

The underlying technique used in [35, 37, 39] was that IS(A) and IS-1(A) axiom systems would apply an essentially 2-part formalism to achieve the above conditions. Their first part would achieve the conditions (1) and (2) by essentially simulating the actions of the axiom system A in a relatively straightforward manner. These axiom systems will satisfy the property (3) by using an analog of a Kleene-like "I am consistent" axiom, similar to the statement $+$ (defined in Section 2).

The main challenge in [35, 37, 39] was to demonstrate that the axiom systems IS(A) and IS-1(A) did not become inconsistent on account of the presence of their final axiom — which declared their own consistencies. In the nomenclature of Section 2, the challenge was to show that the presence of the axiom $+$ did not cause either IS(A) or IS-1(A) to violate Part-B of Section 2's definition of introspective unification.

In particular, let us recall that Section 2 had noted that if an axiom system α is sufficiently strong and we add to it a Kleene-like "I am consistent" axiom, then the resulting new system α^* will be rendered inconsistent because of the presence of a Gödel-like diagonalization proof. The main theorems in [37, 39] showed that this difficulty did not pertain to either IS(A) or the stronger IS-1(A) system because these formalisms treated integer multiplication as a 3-way relation, rather than as a total function.

Our articles [35, 37, 39] of course, raised almost as many as questions as they had settled because an axiom system that fails to recognize multiplication as total is inherently weak. *How useful is it to employ such axiom systems which gain a knowledge* *about their own consistency only by sacrificing the common axiom that multiplication is a total function?* This question is especially pressing because [36, 38] demonstrated it is impossible to construct a consistent axiom system that simultaneously recognizes integer-multiplication as a total function and its own consistency — even when one uses the quite weak Level(0-) definition of tableaux consistency !

Within such a context, we will now show that there is an analog of a computer's floating point instruction set, *called simulated-real arithmetic*, where the axiom systems IS(A) and IS-1(A) from [37, 39] can prove that addition, multiplication, subtraction and division *among simulated real numbers* are total functions.

Definition 1. We will use two formalizations of an integer, called NN and IPN, in this paper. The first definition "NN" will represent the set of non-negative integers. (This is the usual definition employed when investigating the Incompleteness Theorem.) Our alternate definition "IPN" will regard an integer as being any *positive or negative* whole number, as well as reserve a special symbol for representing ∞ .

Definition 2. The symbol F will denote a 1-1 function F that maps the set of NN integers onto IPN integers. In particular, let Even(x) denote a function that equals 1 if x is an even number and -1 if x is odd. Let Half(x) denote the integer-truncated quantity $\lfloor x \div 2 \rfloor$. Then $F(x)$ is defined by the convention that:

$$F(x) \ = \ \text{Even}(x) \cdot \text{Half}(x) \ \text{ when } \ x \neq 1 \qquad \text{AND} \qquad F(1) \ = \ \infty$$

Lower case letters x will henceforth denote NN-integers, and upper case letters X will denote IPN integers.

Definition 3. Let i denote an arbitrary indexing integer. Then the $i-$th **Simulated Real-Number** will be defined to be an ordered pair (M_i, E_i) where M_i is an IPN number storing the mantissa, and E_i is a second IPN integer storing the exponent. The bold-face symbol $\mathbf{R}_i$ will denote this simulated real-number. It is defined as follows:

1. If $E_i \neq \infty$ and $0 \neq M_i \neq \infty$ then $\mathbf{R}_i \ = \ M_i \cdot 2^{\,-\lfloor Log_2(\,|\,M_i\,|\,)\rfloor} \cdot 2^{\,E_i}$.
2. If $E_i = \infty$ and M_i is a power of 2, then $\mathbf{R}_i$ represents the real number 0 written in a binary notation with Log(M_i) digits to the right of the decimal point.
3. Otherwise, $\mathbf{R}_i$ will represent an "overflow" symbol following division by zero.

Important Comment: We will often use the NN notation (m_i, e_i) to denote a simulated real number, instead of IPN. In this case, Definition 2's function F will map (m_i, e_i) onto its IPN counterpart (M_i, E_i) , so as to calculate $\mathbf{R}_i$'s value.

Definition 4. Let $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ denote three simulated real-numbers that are encoded by the respective ordered pairs (m_1, e_1), (m_2, e_2) and (m_3, e_3) when written in the NN-integer notation. Let S denote one of the four arithmetic symbols of $+$, $\times$, $-$ or $\div$. Then $\Theta_S(m_1, e_1, m_2, e_2, m_3, e_3)$ will henceforth denote a formula which states that the two real numbers $\mathbf{R}_1$ and $\mathbf{R}_2$, combined under the arithmetic operation of S , will produce a third simulated real of $\mathbf{R}_3$. More precisely, $\Theta_S(m_1, e_1, m_2, e_2, m_3, e_3)$'s definition will employ the usual computerized floating point hardware rounding convention that $\mathbf{R}_3$'s computed mantissa has a bit-length L equal to the maximum of the lengths for the two input mantissas of $\mathbf{R}_1$ and $\mathbf{R}_2$. It will thus specify $\mathbf{R}_3$ represents the *closest approximation of the combination* of $\mathbf{R}_1$ and $\mathbf{R}_2$ under the operation S that is feasible for a number which has a mantissa-length of L and which uses Definition 3's formula for defining $\mathbf{R}_3$'s value.

Lemma 1. *For each of the four cases where* S *denotes one of the symbols of* $+$ *,* $\times$ *,* $-$ *or* $\div$ *, the predicate* $\Theta_S(m_1, e_1, m_2, e_2, m_3, e_3)$ *has a* Δ_0^* *encoding.*

Proof Sketch: For the cases where S denotes the $+$ or $-$ symbols, it is easy to encode $\Theta_S(m_1, e_1, m_2, e_2, m_3, e_3)$ as a Δ_0^* formula. It is also reasonably routine to encode $\Theta_S(m_1, e_1, m_2, e_2, m_3, e_3)$ as a Σ_1^* formula when S denotes the $\times$ or $\div$ symbol. A much more meticulous analysis in a longer version of this paper will show that this formal encoding for multiplication and division can, in fact, be compressed into a more ideally terse Δ_0^* form. The intuition behind this compression is for one to break the initial mantissas m_1 and m_2 into two substrings of essentially equal length, each, so that they are sufficiently short so that the Σ_1^* formula's *unbounded existential quantifiers* can then be made bounded. $\Box$.

Lemma 2. *Let* S *again denote one of the four arithmetic symbols of* $+$ *,* $\times$ *,* $-$ *or* $\div$ *under simulated real arithmetic. In each of these four cases using the NN-integer notation, the statement that* S *is a total function can be encoded as a* Π_1^* *sentence.*

Proof: It is clear that if the statement of Lemma 2 was changed so that the totality of S was expressed as a Π_2^* (rather than Π_1^*) sentence, then Lemma 2 would be an immediate consequence of Lemma 1. This is because in each of the four cases where S denotes the symbol of $+$, $\times$, $-$ or $\div$, Equation (8) is a formal statement declaring the totality of the operation of S :

$$\forall\, m_1 \,\forall\, e_1 \,\forall\, m_2 \,\forall\, e_2 \,\exists\, m_3 \,\exists\, e_3 \quad \Theta_S(m_1, e_1, m_2, e_2, m_3, e_3) \tag{8}$$

In order to construct a Π_1^* sentence that is equivalent to (8) under the Standard Model, we will use the fact that in each of the four cases where S denotes the symbol of $+$, $\times$, $-$ or $\div$, a 6-tuple will satisfy $\Theta_S(m_1, e_1, m_2, e_2, m_3, e_3)$ only when:

$$** \quad m_3 \,\leq\, \text{Double(Max}(m_1, m_2)) \;\text{ AND }\; e_3 \,\leq\, \text{Double(Double(Max}(e_1, e_2)))$$

Let t denote the term of Double(Double(Max(m_1, m_2, e_1, e_2))). Item $**$ then implies that Equation (9) is a Π_1^* sentence that logically implies the validity of Equation (8) under the Standard Model of the Natural Numbers:

$$\forall\, m_1 \;\forall\, e_1 \;\forall\, m_2 \;\forall\, e_2 \;\exists\, m_3 \,\leq\, t \;\exists\, e_3 \,\leq\, t \quad \Theta_S(m_1, e_1, m_2, e_2, m_3, e_3) \tag{9}$$

Definition 5. In a context where $\mathbf{R}$ denotes a simulated real number, the term Expand($\mathbf{R}$) will denote a second simulated real whose value is identical to that of $\mathbf{R}$ except that the mantissa for Expand($\mathbf{R}$) will have one extra bit of precision. Thus, if $\mathbf{R}_1 = (m_1, e_1)$ and $\mathbf{R}_2 = (m_2, e_2)$ then Equation (10) is a Δ_0^* formula, denoted formally as $\Theta^*(m_1, e_1, m_2, e_2)$, which indicates that $\mathbf{R}_2$ = Expand($\mathbf{R}_1$) .

$$m_2 \;=\; \text{Double}(m_1) \;-\; \text{Count}(m_1, 1) \quad \text{AND} \quad e_1 \;=\; e_2 \tag{10}$$

Lemma 3. *There exists a Π_1^* formula indicating that Expand$(\mathbf{R})$ is a total function.*

Proof: Obvious because the Equation (11) is the needed formula:

$$\forall\, m_1 \;\; \forall\, e_1 \;\; \exists\, m_2 \leq \text{Double}(m_1) \;\; \exists\, e_2 \leq e_1 \qquad \Theta^*(m_1, e_1, m_2, e_2) \tag{11}$$

Theorem 3. (An initial result that will be made much more robust in Section 5): *For every consistent axiom system A employing the U-Grounding functions, there exists a consistent axiom system α that can $1)$ prove all A's Π_1^* theorems, $2)$ recognize integer-addition as a total function, $3)$ confirm that simulated real arithmetic operations of addition, multiplication, subtraction, division and Expand are each total functions, and $4)$ recognize its own Level(1) semantic tableaux consistency.*

Proof: Let Ψ_1, Ψ_2, Ψ_3, Ψ_4 and Ψ_5 denote the five Π_1^* sentences, defined by Lemmas 2 and 3 that indicate the operations of addition, multiplication, subtraction, division and Expand are each total functions. Let A' denote the union of the axiom system A with these five added Π_1^* sentences of Ψ_1, Ψ_2, Ψ_3, Ψ_4 and Ψ_5. It follows from the combination of Theorem 2 and Lemmas 2 and 3 that the axiom system IS-1(A') satisfies Theorem 3's four requirements $\square$

The remainder of this article will have two parts. Section 4 will introduce two new versions of the Second Incompleteness Theorem, whose negative results *tightly contrast against* Theorem 3's positive result. Section 5 will introduce a stronger form of Theorem 3, which replaces the notion of Level(1) tableaux consistency with the significantly stronger construct of "$Tier(1)$" tableaux consistency. It will also contemplate potential applications to numerical analysis. It will arrive at a pleasing mathematical observation about the foundational nature of Gentzen-style deductive cuts.

4 Generalizations of the Second Incompleteness Theorem

The term **"Base"** substructure will refer to the starting axiom system A which Theorem 2 used to construct its more elaborate self-justifying axiom systems of IS(A) and IS-1(A). It is apparent that Theorem 3's partial exception to the Second Incompleteness Theorem possesses some type of non-trivial quality because its axiom systems of IS(A) and IS-1(A) have a capacity to prove all the Π_1^* theorems of Peano Arithmetic (PA) whenever $A \supset PA.$ Since the Second Incompleteness Theorem precludes a self-justifying axiom system α from becoming excessively strong, it is thus helpful to remind ourselves about the exact properties which α *is precluded from possessing.* For example, Items I and II from the literature survey chapter indicated Type-S axiomizations *of integer arithmetic* cannot recognize their own Hilbert consistency [20, 26], and similarly Type-M axiomizations *of integer arithmetic* cannot recognize their Level(0-) tableaux consistency [38]. Also in another recent article [40], we established Type-A axiomizations *of integer arithmetic* are unable to recognize their own Level(2) tableaux consistency. In this section, we will develop two variants of the Second Incompleteness Theorem that apply uniquely to *simulated real arithmetics.*

Definition 6. Let **AddComp**$(m_1, e_1, m_2, e_2, m_3, e_3)$ denote a Δ_0^- *predicate formula* indicating that (m_3, e_3) represents the additive sum of the simulated real numbers of (m_1, e_1) and (m_2, e_2) that satisfies the usual computerized hardware-rounding convention that the mantissa m_3 has a bit-length equal to the maximum of the bit-lengths for m_1 and m_2. Also, let **LongMult**$(m_1, e_1, m_2, e_2, m_3, e_3)$ denote a Δ_0^- predicate formula indicating that (m_3, e_3) represents the *untruncated* multiplicative product of the simulated real numbers of (m_1, e_1) and (m_2, e_2) . (This is the variant of multiplication where the floating-point truncate-and-round operation is now absent.)

The first of our two new versions of the Second Incompleteness Theorem is Theorem 4. It will show that Theorem 3's partial evasion of the Second Incompleteness Theorem has no analog when one tries to generalize it from semantic tableaux styled definitions of consistency to stronger definitions of consistency focused around Hilbert deduction. Theorem 4's result is surprising because semantic tableaux and Hilbert deduction produce the same set of theorems in first order logic. The intuitive reason that Theorems 3 and 4 provide such a contrasting pair of results is because these two formalisms produce sharply different proof lengths.

Theorem 4. *There exists a Π_1^- sentence W (with a relatively simple structure) such that every consistent Grounding-language based axiom system $\alpha \supset W$* **is unable to both** *prove its own consistency (under the Hilbert-styled method of deduction) and to also prove Equation (12)'s Π_2^- sentence.* (The latter states AddComp formalizes a total function among simulated real numbers).

$$\forall\, m_1 \;\forall\, e_1 \;\forall\, m_2 \;\forall\, e_2 \;\exists\, m_3 \;\exists\, e_3 \quad \text{AddComp}(m_1, e_1, m_2, e_2, m_3, e_3) \tag{12}$$

Proof Sketch: We shall employ the Theorem 1 by Pudlák-Solovay, which had indicated that *no axiom system can simultaneously* recognize successor as a total function and prove a theorem affirming its own Hilbert consistency. Our current discussion cannot assume *apriori* that Theorem 4's axiom system α will recognize successor as a total function — because α is employing a Grounding (rather than U-Grounding) language *without growth functions.* Thus to prove Theorem 4, we will need to show that α can infer from Equation (12) that successor is a total function.

Let us first recall that our notation convention has the upper case symbol X replacing the lower case symbol x when we are viewing an integer object in the IPN rather than the NN notation. Also, let $\bar{C}$ be a representation for the constant " $+1$ " written in IPN notation. Using this notation to simplify (12) and letting $\bar{C}$ substitute for m_1 and m_2 , it follows that the axiom system α can infer (13) from (12).

$$\forall\, E_1 \;\exists\, M_3 \;\exists\, E_3 \quad \text{AddComp}(\, \bar{C}\, ,\, E_1\, ,\, \bar{C}\, ,\, E_1\, ,\, M_3\, ,\, E_3\,) \tag{13}$$

A longer version of this paper will use Equation (13) to derive that if our threshold Π_1^* sentence W contains more than some tiny amount of strength then any $\alpha \supset W$ which proves Equation (12)'s validity shall be able to infer from (13) that successor is a total function among the set of NN integers. Also assuming that W contains more than a tiny amount of initial strength, α will then satisfy Theorem 1's hypothesis. Thus α cannot verify its own Hilbert consistency. □

Theorem 5. *There exists some particular Π_1^- sentence W such that no consistent axiom system $\alpha \supset W$ can simultaneously prove its own Level(0-) tableaux consistency and the validity of Equation (14)'s Π_2^- sentence.* (Equation (14) indicates that LongMult formalizes a total function among simulated real numbers).

$$\forall\, m_1 \;\forall\, e_1 \;\forall\, m_2 \;\forall\, e_2 \;\exists\, m_3 \;\exists\, e_3 \quad \text{LongMult}(m_1, e_1, m_2, e_2, m_3, e_3) \tag{14}$$

Proof Sketch: Our justification of Theorem 5 will be brief because its proof is similar to Theorem 4's proof. After one changes the underlying notation from NN to IPN, it is apparent that LongMult$(M_1, E_1, M_2, E_2, M_3, E_3)$ can be satisfied only when $M_3 = M_1 \cdot M_2$. Hence, (14) implies multiplication is a total function among IPN integers. Also, there clearly exists a Π_1^- sentence Φ that can infer multiplication is a total function *among NN integers* from its totality *among IPN integers.*

Let us next recall that our Tableaux-2000 conference paper [36] (and/or its journal counterpart [38]) constructed a Π_1^- sentence V where no Type-M *integer-based* system $\alpha \supset V$ can prove its Level(0-) tableaux consistency. The analog of this result for simulated real arithmetics thus follows by setting $W = V \cup \Phi$. □

On the Significance of Theorems 4 and 5: We suspect that Theorem 4 is the more important of the two restrictions upon the capacities of simulated real systems. One reason Theorem 5 is less significant is because digital computers simply do not process floating point numbers with the LongMult instruction set. (If they did, then the bit-length of a floating point number would double each time a multiplication was performed: It would exceed the number of atoms in the universe after roughly 100 consecutive multiplications.) It is thus reassuring that numerical analysts, starting with Leibnitz and Newton, have demonstrated most mathematical computations are *only infinitesimally changed* when one one throws away their low-order bits.

On the other hand, Theorem 4 formalizes a *more serious* constraint on self-justifying systems. It signals a Gentzen-style deductive cut rule will usher in the force of the Second Incompleteness Theorem. Thus, we are left to inquire: *In what respects, beyond those delineated by Theorem 3, can Theorem 4's actual generalization of the Second Incompleteness Theorem be evaded? Also how robust may such evasions be?* The next section will contain our chief results. It will bring this article to a surprising conclusion.

5 On the Two Surprising Facets of Hybrid Deduction Formalisms

Let us say a U-Grounding sentence belongs to the Tier(k) class if it is either Π_k^* or Σ_k^*. Also, let H denote a sequence of ordered pairs (t_1, p_1), (t_2, p_2), ... (t_n, p_n) , where p_i is a semantic tableaux proof of the theorem t_i . In a context where $\Re$ designates some class of sentences, such as possibly Tier(k), Π_k^* or Σ_k^*, define H to be a Tab$-\Re$ proof of a theorem T from the axiom system α iff $T = t_n$ and also:

1. Each axiom in p_i's proof is either one of $t_1, t_2, ...t_{i-1}$ or comes from α.
2. Each of the "intermediate results" $t_1, t_2, ...t_{n-1}$ lie in the pre-specified class $\Re$.

Thus, Tab$-\Re$ deduction is stronger than classic tableaux by allowing for a type of Gentzen-like deductive cut rule for sentences that belong to the intermediate class, that is formalized by $\Re$. From Gentzen's Cut Elimination Theorem [28] it is known that the set of the theorems that can be proven by a Tab$-\Re$ proof *are the same* as those that can be derived by a purely cut-free proof formalism, similar to semantic tableaux. However, the proof length efficiencies of these two alternate approaches *can be quite different.*

If $\Re$ denotes the universal class of all possible sentences then Tab$-\Re$ deduction will thus be essentially equivalent to the Hilbert-style deductive method in its proof-efficiency. On the other hand, if $\Re$ denotes a class of sentences that is a strict subset of the universal class, such as for example Tier(k), then Tab$-\Re$ will be a methodology lying properly between classic tableaux and Hilbert deduction.

This terminology was introduced in [40]. It proved Type-A axiom systems are unable to verify their self-consistency under either Tab$-\Sigma_2^*$ or Tab$-\Pi_2^*$ deduction.

The acronym Tab-1 will refer to a version of Tab$-\Re$ deduction with $\Re$ = Tier(1). An expanded version of our Tableaux-2002 paper [39], which will soon appear in the *Journal of Symbolic Logic* [41], will explore a stronger variant of [39]'s IS-1(A) formalism, called IS$_D(A)$. Their difference is that IS$_D(A)$ can recognize its consistency under Tab-1 deduction, while IS-1(A) recognized its consistency under only Section 3's Level(1) definition. (Both formalisms prove all A's Π_1^* theorems.) The last page of our Tableaux-2002 paper actually defined IS$_D(A)$ and called it "IS-1*(A)". The longer JSL article [41] will finish this topic by formally proving IS$_D(A)$ is consistent.

Theorem 6. (A stronger version of Theorem 3) *For every consistent axiom system A employing the U-Grounding functions, there exists a consistent axiom system α that can* $1)$ *prove all A's Π_1^* theorems,* $2)$ *recognize integer-addition as a total function,* $3)$ *confirm that the five basic operations of simulated real arithmetic are total functions,* **and** $4)$ **recognize its own consistency under Tab-1 deduction.**

Proof Sketch: The proof for Theorem 6 is identical to Theorem 3's proof, except that all references to the prior paper [39]'s IS-1 system should be replaced by the forthcoming paper [41]'s IS$_D$ system to achieve Feature (4)'s added functionality. □

IMPORTANT COMMENT: Since Theorem 6 is a direct generalization of Theorem 3, it is tempting to suspect both results have similar implications. *To the contrary,* we will now show Theorem 6 has a special added *"Tier$(1)^{\oplus}$ floating-point"* property, which is actually central for achieving our main goals and purposes.

NOTATION: The symbol $\langle m, e\rangle$ will denote the simulated real number with mantissa m and exponent e. Also $|\langle m, e\rangle|^J$ will denote the quantity begotten by taking the absolute value of this simulated real number and raising it to the $J-$th power. Assuming that $J \neq 0$, $\langle n, f\rangle \geq 1$ and that Length(m) denotes m's bit-length, the formal expression of $\langle m, e\rangle \ll_L^J \langle n, e\rangle$ will have the following meaning:

1. Length$(m) \leq$ Length$(n) + L$ and if $J \geq 1$ then $|\langle n, f\rangle|^J \geq |\langle m, e\rangle| \geq 1.$
2. Length$(m) \leq$ Length$(n) + L$ and if $J \leq -1$ then $|\langle n, f\rangle|^J \leq |\langle m, e\rangle| \leq 1.$

Assuming that **R** is a term that specifies the value of a simulated real number whose value is greater than 1, we will also use the preceding notation to define **Bounded**

Real Quantifiers of the form $\exists \langle m,e\rangle \ll_L^J$ **R** and $\forall \langle m,e\rangle \ll_L^J$ **R** . The term **Bounded Integer Quantifiers** will refer to the expressions of the form $\forall x \leq t$ and $\exists x \leq t$ that were defined in Section 3. A wff will be called $\Delta_0^{\oplus}$ if it is built in any arbitrary manner out of these four forms of bounded quantifiers, together with the usual U-Grounding function symbols, the equality and greater-than predicates and the standard Boolean connectives. If Ψ is a $\Delta_0^{\oplus}$ formula then the formal expressions of $\forall v_1 \; \forall v_2 \; ... \forall v_k \, \Psi$ and $\exists v_1 \; \exists v_2 \; ... \exists v_k \, \Psi$ will be called $\Pi_1^{\oplus}$ and $\Sigma_1^{\oplus}$ formulae. Also a sentence will be called Tier$(1)^{\oplus}$ when it is either $\Pi_1^{\oplus}$ or $\Sigma_1^{\oplus}$.

Theorem 7. *There exists a computable function F that maps each Tier$(1)^{\oplus}$ formula ϕ onto a Tier(1) formula Φ such that ϕ and Φ are logically equivalent to each other.*

The theory of LinH functions [43] can be used to prove Theorem 7. This proof is rather lengthy. It will thus appear in a longer version of this article.

On the Surprising Combined Implications of Theorems 3, 4, 6 and 7: Theorem 4 had showed that Theorem 3's evasion of the Second Incompleteness Effect for tableaux deduction does not generalize for Hilbert deduction. In this context, Theorem 6 had offered an alternate hybrid approach, called Tab-1 deduction, for at least partially extending the prior tableaux results. One subsequent question is therefore to ask: *How much more robust is Tab-1 deduction than classic tableaux ?*

In order to answer this question, let Φ denote a Tier(1) sentence of either the form of $\forall v_1 \; \forall v_2 \; .. \forall v_k \, \Psi$ or $\exists v_1 \; \exists v_2 \; .. \exists v_k \, \Psi$, and let us call Ψ the Δ_0^* **Stem** of Φ. Since by definition, all the quantifiers in the stem are bounded integer quantifiers of the form $\forall x \leq t$ or $\exists x \leq t$, it follows that the size of the integer x is greatly limited by the size of the input integers $p_1, p_2 ... p_j$ for the term t. In particular since the only growth functions available in our U-Grounding language are addition and doubling, the size of x may exceed Max$(p_1, p_2 ... p_j)$ by only a scalar constant of k — whose value depends on the number of function symbols in t . The latter fact shows that the IS$_D(A)$ self-justifying formalism can actually prove only a *limited range* of theorems about *"integer"* arithmetics because IS$_D(A)$ is able to include only the Π_1^* theorems of A as its generating set of axioms, and it can apply a Gentzen-style deductive cut rule *only to the quite limited Tier(1) class of formulae.*

However if one *shifts the venue of application* from integer arithmetic to *real-value arithmetic*, then the range of permissible uses of a Gentzen-style deductive cut rule becomes much more robust. This is because Theorem 7 stated that every Tier$(1)^{\oplus}$ formula ϕ can be translated into an equivalent Tier(1) formula Φ. Moreover, the bounded real quantifiers in a Tier$(1)^{\oplus}$ formula of the form $\exists \langle m,e\rangle \ll_L^J$ **R** and $\forall \langle m,e\rangle \ll_L^J$ **R** allow $\langle m,e\rangle$'s real number to attain values essentially as large as $\mathbf{R}^J$ (quite unlike the narrower range of values associated with integer-based bounded quantifiers). Thus *from the perspective of real valued arithmetic,* the IS$_D(A)$ axiom has a good deal of flexibility, since any logically valid $\Pi_1^{\oplus}$ sentence can essentially be made into an axiom (by choosing an initial broad enough base axiom system of A) , *and the Tab-1 deductive cut rule can then be applied to any Tier$(1)^{\oplus}$ formula.*

Hence, while self-justifying systems may be viewed as primarily a *theoretical-only* device for exploring *integer arithmetics*, their significance is broader for real-valued arithmetics *because of the permissible use of bounded real quantifiers.*

To reinforce this point, we observe that all the major algorithms in numerical analysis [7] can be simulated by proofs under simulated real arithmetic. The basic purpose of numerical analysis is to produce sequences of real numbers $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \ldots$ that converge upon target answers with a decreasing error rate $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$. By choosing to use mantissas with sufficiently large lengths, it is easy to formalize such algorithms under simulated real arithmetic. Each of the axiom systems of IS(A), IS-1(A) and IS$_D(A)$ can accomplish this task. Their distinction is that they house the desired simulations under three increasingly broad definitions of tableaux self-consistency.

The seminal nature of Gödel's Second Incompleteness Theorem is, of course, impossible to overestimate. It has had many fascinating generalizations. Our exceptions to Gödel's Second Incompleteness Theorem illustrate that there are certain *limited-but-tangible* respects where a formalism *not using the integer version of multiplication*, can possess a *partial-but-not-full* knowledge of its own consistency.

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References

1. Z. Adamowicz, "Herbrand Consistency and Bounded Arithmetic", *Fundamenta Mathematica* 171 (2002) pp. 279-292.
2. Z. Adamowicz and T. Bigorajska, "Existentially Closed Structures and Gödel's Second Incompleteness Theorem", *Journal of Symbolic Logic* 66 (2001), pp. 349-356.
3. Z. Adamowicz and P. Zbierski, *The Logic of Mathematics*, John Wiley and Sons, 1997.
4. Z. Adamowicz and P. Zbierski, "On Herbrand consistency in weak theories", *Archive for Mathematical Logic* 40 (2001) pp. 399-413.
5. T. Arai, "Derivability Conditions on Rosser's Proof Predicates", *Notre Dame Journal on Formal Logic* 31 (1990) pp. 487-497.
6. A. Bezboruah and J. Shepherdson, "Gödel's Second Incompleteness Theorem for Q", *Journal of Symb Logic* 41 (1976) 503-512.
7. R. Burden and J. Faires, *Numerical Methods*, Brookes-Cole (2003)
8. S. Buss and A. Ignjatovic, "Unprovability of Consistency Statements in Fragments of Bounded Arithmetic", *Annals of Pure and Applied Logic* 74 (1995) pp. 221-244.
9. M. Fitting, *First Order Logic & Automated Theorem Proving,* SpringerVerlag 1996.
10. K. Gödel, " Über formal unentscheidbare Sätse der Principia Mathematica und Verwandte Systeme I", *Monatshefte für Math. Phys.* 37 (1931) pp. 349-360.
11. P. Hájek, "On Interpretability in Set Theory" Part I in *Com. Math. Univ. Carol* 12 (1971) pp. 73–79 and Part II in *Comm. Math. Univ. Carol* 13 (1972) pp 445-455.
12. P. Hájek and P. Pudlák, *Metamathematics of First Order Arithmetic*, Springer Verlag 1991.
13. R. Jeroslow, "Consistency Statements in Formalal Mathematics.", *Fundamenta Mathemtaica* 72 (1971) pp. 17-40.
14. S. Kleene, "On the Notation of Ordinal Numbers", *Journal Symb Logic* 3 (1938), 150-156.
15. J. Krajícek, "A Note on Proofs of Falsehood", *Archive for Math Logic* 26 (1987) 169-176.
16. G. Kreisel and G. Takeuti, "Formally Self-Referential Propositions for Cut-Free Classical Analysis", *Dissertationes Mathematicae* 118 (1974) pp. 1–55
17. E. Nelson, *Predicative Arithmetic,* Princeton Math Notes, 1986.
18. J. Paris and C. Dimitracopoulos, "A Note on the Undefinability of Cuts", *Journal of Symbolic Logic* 48 (1983) pp. 564-569.

19. J. Paris and A. Wilkie, "Δ_0 Sets and Induction", *1981 Jadswin Conf Proc*, pp. 237-248.
20. P. Pudlák, "Cuts, Consistency Statements and Interpretations", *Journal of Symbolic Logic* 50 (1985) 423-442.
21. P. Pudlák, "On the Lengths of Proofs of Consistency", in *Collegium Logicum: 1996 Annals of the Kurt Gödel Society* (Volume 2), Springer-Wien-NewYork, pp 65-86.
22. H. Rogers Jr. , *Recursive Functions and Effective Compatibility,* McGrawHill 1967.
23. J. Rosser, "Extensions of Theorems by Gödel and Church", *Jour Symb Logic* 1 (1936) 87-91.
24. S. Salehi, Ph D thesis for the Polish Academy of Sciences, Oct. 2001.
25. C. Smoryński, "Non-standard Models and Related Developments in the Work of Harvey Friedman", in *Harvey Friedman's Research in Foundations of Math.*, North Holland 1985.
26. R. Solovay, Private telephone communications during 1994 describing Solovay's generalization of one of Pudlák's theorems [20], using the additional formalisms of Nelson and Wilkie-Paris [17, 34]. Solovay never published this result (called Theorem 1 in Section 2) or his other observations that several logicians [12, 17–21, 34] have attributed to his private communications. The Appendix A of [37] offers a 4-page summary of Solovay's theorem.
27. V. Švejdar, "Modal Analysis of Generalized Rosser Sentences", *Journal of Symbolic Logic* 48 (1983) 986-999.
28. G. Takeuti, *Proof Theory,* Studies in Logic Volume 81, North Holland, 1987.
29. G. Takeuti, "Gödel Sentences of Bounded Arithmetic", *Journal of Symbolic Logic* 65 (2000) pp. 1338-1346
30. A. Tarski, A. Mostowski and R. M. Robinson, *Undecidable Theories,* North Holland, 1953.
31. A. Visser, "The Unprovability of Small Inconsistency", *Archive for Mathematical Logic* 32 (1993) pp. 275-298.
32. A. Visser, "Faith and Falsity", to appear in *Annals of Pure and Applied Logic.*
33. P. Vopěnka and P. Hájek, "Existence of a Generalized Semantic Model of Gödel-Bernays Set Theory", *Bulletin de l'Academie Polonaise des Sciences* 12 (1973) pp.1079-1086.
34. A. Wilkie and J. Paris, "On the Scheme of Induction for Bounded Arithmetic", *Annals of Pure and Applied Logic* (35) 1987, 261-302
35. D. Willard, "Self-Verifying Axiom Systems", *Proceedings of the Third Kurt Gödel Colloquium* (1993), Springer-Verlag LNCS#713, pp. 325-336.
36. D. Willard, "The Semantic Tableaux Version of the Second Incompleteness Theorem Extends Almost to Robinson's Arithmetic Q", *Proceedings of the Tableaux 2000 Conference*, SpringerVerlag LNAI#1847, pp. 415-430.
37. D. Willard, "Self-Verifying Systems, the Incompleteness Theorem and the Tangibility Principle", in *Journal of Symbolic Logic* 66 (2001) pp. 536-596.
38. D. Willard, "How to Extend The Semantic Tableaux And Cut-Free Versions of the Second Incompleteness Theorem Almost to Robinson's Arithmetic Q", in *Journal of Symbolic Logic* 67 (2002) pp. 465–496.
39. D. Willard, "Some Exceptions for the Semantic Tableaux Version of the Second Incompleteness Theorem", *Proceedings of the Tableaux 2002 Conference*, SpringerVerlag LNAI#2381, pp. 281–297.
40. D. Willard, "A Version of the Second Incompleteness Theorem For Axiom Systems that Recognize Addition But Not Multiplication as a Total Function", *First Order Logic Revisited, (Year 2003 Proceedings FOL-75 Conference),* Logos Verlag (Berlin) 2004, pp. 337–368.
41. D. Willard, "An Exploration of the Partial Respects in which an Axiom System Recognizing Solely Addition as a Total Function Can Verify Its Own Consistency", to appear in the *Journal of Symbolic Logic*.
42. D. Willard, "A New Variant of Hilbert Styled Generalization of the Second Incompleteness Theorem and Some Exceptions to It", to appear in the *Annals of Pure and Applied Logic*.
43. C. Wrathall, "Rudimentary Predicates and Relative Computation", *Siam Journal on Computing* 7 (1978), 194-209