Practice Problems

1. Formally prove that the following triples are valid:
   
   (a) \{x \neq 0\} \ x = x \times x; \ {x > 0}\)

   (b) \{|x| > 2\} if (x < 0) \{x = -x;\} else \{x = x + 1;\} \{x \neq 2\}

   Using the proof rule for if-then-else, split into 2 triples

   \{|x| > 2, x < 0\} x = -x; \{(x \neq 2)\} and

   \{|x| > 2, x \geq 0\} x = x + 1; \{(x \neq 2)\}.

   Now apply the assignment rule to each.

   (c) \{y > x\} \ t = x; \ x = y; \ y = t; \{x > y\}

   (d) \{u == x + y \times t\} \ t = t + 1; \ x = x - y; \{u == x + y \times t\}

   In both (c) and (d) the proofs will be easier if you use the weakest liberal preconditions. For instance, in (c),

   \text{wlp}(y = t, (x > y)) = (x > t)

   \text{wlp}(x = y, (x > t)) = (y > t)

   \text{wlp}(t = x, (y > t)) = (y > x)

2. Choose the correct invariants to prove (formally) that the following triples are valid:

   (a) \{(x == 1) \land (S == 0) \land (y \geq x)\}

       \text{while } (x <= y) \{S = S + x; \ x = x + 1;\}

       \{S == y \times (y + 1)/2\}

       Use I = \{(x <= y+1), S == x(x-1)/2\}
(b) \{2x > y > x > 0\} \text{ while } ((y \% x) > 0) \{x = x + 1;\} \{x == y\}

This is trickier. The following invariant will work:

\{(2x > y >= x > 0), ((y \% x) >= 0)\}

Actually the second part, \((y \% x) >= 0\), is not strictly necessary since this is always true for positive integers \(x\) and \(y\).

3. Formally verify the following program using axiomatic semantics. \(U\) and \(V\) are the inputs and \(S\) is the output. Both \(U\) and \(V\) are non-negative integers.

\begin{verbatim}
P: \{U \geq 0, V \geq 0\}
x = U;
y = V;
S = 0;

while \(x > 0\)
    \{if ((x \% 2) == 0)
        \{x = x / 2;
        y = y * 2;
    }
    else
        \{x = x - 1;
        S = S + y;
    \}
\}

Q: \{S == UV\}

Use
\{x \geq 0 \land xy + S == UV\}

as the loop invariant.