1. State whether the following quantified formulae are true over the natural numbers \( \mathbb{N} = \{1, 2, \ldots \} \):

(i) \( \forall x \exists y \forall z \ [xy \neq z^2] \)

**True:** Take \( y = 2x \).

(ii) \( \exists x \forall y \exists z \ [(x = y^2) \lor (y = z^2)] \).

**False:** Take \( y = 2x^2 \).

(iii) \( \forall x \exists y \forall z \exists w \ [\left| x - z \right| \geq \left| y - w \right|] \)

**True:** Take \( y = x \) and \( w = z \).

2. Use induction on \( n \) to prove that for all \( n \geq 2 \), \( 2^n + 3^n < 5^n \).

Clearly \( 2^2 + 3^2 = 13 < 25 = 5^2 \). If \( 2^k + 3^k < 5^k \), then \( 2^{k+1} + 3^{k+1} < 5^{k+1} \), since \( 2^{k+1} < 5 \times 2^k \), \( 3^{k+1} < 5 \times 3^k \) and thus

\[
2^{k+1} + 3^{k+1} < 5 \times (2^k + 3^k) < 5^{k+1}.
\]

3. Let \( n \) be a natural number. Show that any set \( S \) of natural numbers of cardinality \( n \) (i.e., \( |S| = n \)) has a subset \( S' \) the sum of whose elements is a multiple of \( n \). (For instance, consider the set \( \{3, 11, 13, 18, 21\} \). Elements of the subset \( \{3, 11, 21\} \) add up to 35 which is a multiple of 5.)

Let \( S = \{a_1, \ldots, a_n\} \). Consider the chain of subsets

\[
\emptyset \subset \{a_1\} \subset \{a_1, a_2\} \subset \ldots \subset \{a_1, \ldots, a_{n-1}\} \subset S
\]

and consider the sums of elements of each of these sets. Since there are \( n + 1 \) sets in this chain there must be two, say \( A \) and \( B \), such that \( A \subset B \) and their sums are the same modulo \( n \). Then the sum of the elements in \( B \setminus A \) must be 0 modulo \( n \).