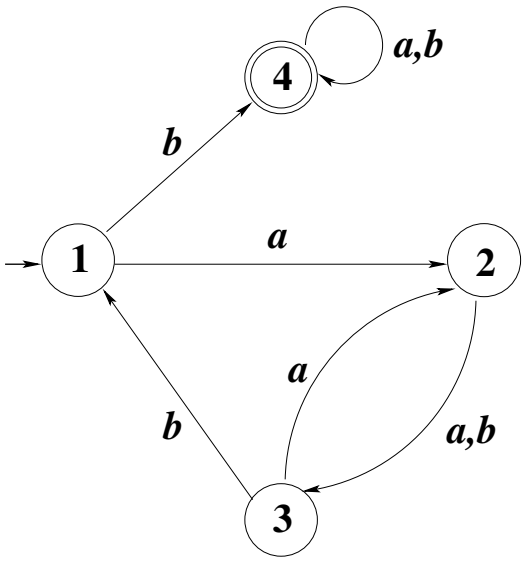
CSI 409 — Fall 2017: Midterm Exam #II
Some answers and hints

1. Derive a regular expression for the language recognized by the following DFA:



The equations are

$$A_1 = aA_2 \cup bA_4 \quad (1)$$
$$A_2 = (a \cup b)A_3 \quad (2)$$
$$A_3 = aA_2 \cup bA_1 \quad (3)$$
$$A_4 = (a \cup b)A_4 \cup \varepsilon \quad (4)$$

By Arden's Lemma, $A_4 = (a \cup b)^*$ and thus equation (1) becomes $A_1 = aA_2 \cup b(a \cup b)^*$. Replacing A_2 in (3), we get

$$A_3 = (aa \cup ab)A_3 \cup bA_1$$

Thus $A_3 = (aa \cup ab)^* bA_1$ and $A_2 = (a \cup b)(aa \cup ab)^* bA_1$. Finally replacing in (1), we get

$$A_1 = a(a \cup b)(aa \cup ab)^* bA_1 \cup b(a \cup b)^*$$

Applying Arden's Lemma,

$$A_1 = \big((aa \cup ab)(aa \cup ab)^* b\big)^* b(a \cup b)^*$$

or

$$A_1 = \big((aa \cup ab)^+ b\big)^* b(a \cup b)^*$$

2. (This question has 3 parts.)

Consider the following language over the alphabet $\{a, b\}$:

$$\{ a^m b^n \mid m > n^2 \geq 0 \}$$

(a) (1 point) Exhibit a string of length 6 that belongs to the above language.

a^6

(b) (5 points) Exhibit a string over $a^+ b^+$ that does **not** belong to the above language.

ab

(c) (28 points) Prove that the above language is not regular.

Pick $w = a^{p^2+1} b^p$ and remove the pump.

3. Exhibit a context-free grammar for the language denoted by the regular expression

$$\left(ba^* b\right)^* \ \cup \ a$$

The alphabet is $\{a, b\}$.

$$\begin{array}{lcl} S & \rightarrow & XS \mid a \mid \varepsilon \\ X & \rightarrow & bAb \\ A & \rightarrow & aA \mid \varepsilon \end{array}$$

4. Show that the following context-free grammar is ambiguous

$$S \ \rightarrow \ aSSb \ \mid \ ab \ \mid \ ba$$

by exhibiting a string in the language that has two *distinct* derivation trees.

The string $a(ab)^4 b$ has two distinct derivation trees.

5. Convert the grammar

$$S \to SSb \mid aSb \mid \varepsilon$$

to Chomsky Normal Form. Show steps clearly.

(a) "Add $S_0 \to S$"

$$\begin{aligned} S_0 &\to S \\ S &\to SSb \mid aSb \mid \varepsilon \end{aligned}$$

(b) "Chop down long rules"

$$\begin{aligned} S_0 &\to S \\ S &\to SZ_1 \mid aZ_2 \mid \varepsilon \\ Z_1 &\to Sb \\ Z_2 &\to Sb \end{aligned}$$

(c) "Eliminate ε-rules"

There are 2 nullable variables: S, S_0

$$\begin{aligned} S_0 &\to S \mid \varepsilon \\ S &\to SZ_1 \mid aZ_2 \mid Z_1 \\ Z_1 &\to Sb \mid b \\ Z_2 &\to Sb \mid b \end{aligned}$$

(d) "Eliminate unit rules"

$$\begin{aligned} S_0 &\to SZ_1 \mid aZ_2 \mid Sb \mid b \mid \varepsilon \\ S &\to SZ_1 \mid aZ_2 \mid Sb \mid b \\ Z_1 &\to Sb \mid b \\ Z_2 &\to Sb \mid b \end{aligned}$$

(e) "Fix right-hand sides of length 2"

$$\begin{aligned} S_0 &\to SZ_1 \mid X_aZ_2 \mid SX_b \mid b \mid \varepsilon \\ S &\to SZ_1 \mid X_aZ_2 \mid SX_b \mid b \\ Z_1 &\to SX_b \mid b \\ Z_2 &\to SX_b \mid b \\ X_b &\to b \\ X_a &\to a \end{aligned}$$