403: Algorithms and Data Structures

A Quick Introduction to Graphs and Trees

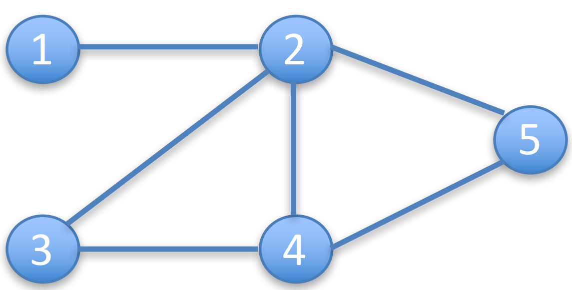
Fall 2016

UAlbany

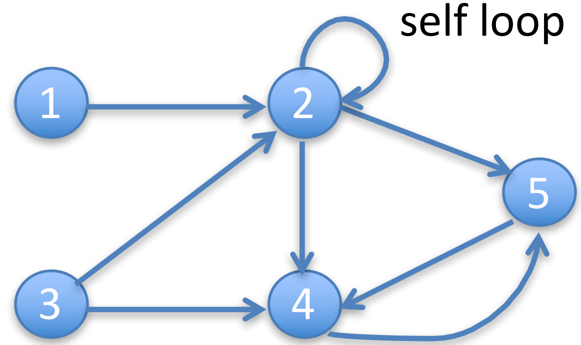
Computer Science

Graphs

- A graph G(V,E) consists of
  - Set of nodes (vertices) V
  - Set of edges E connecting nodes
- Graphs are extremely useful in modeling problems
- Directed graphs represent binary relations in a set (the nodes). Undirected: symmetric binary relation

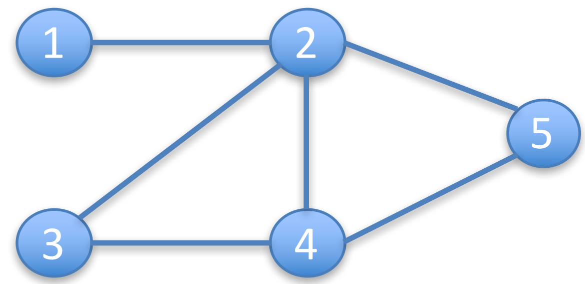


(Undirected) graph



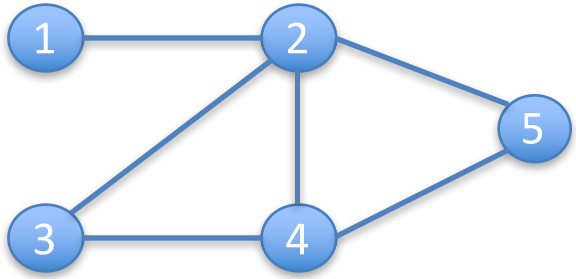
Directed graph (Digraph)

Undirected graphs (terminology)



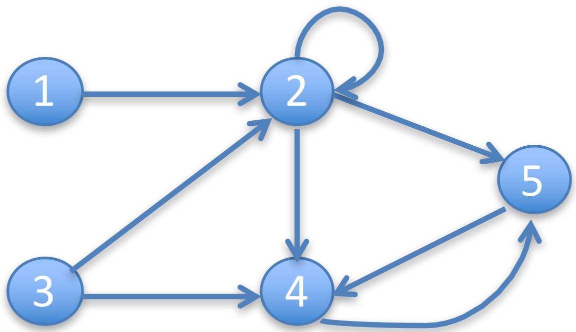
- The edge between 1 and 2 is denoted as (1,2)
  - 1 is **adjacent** to 2
  - (1,2) is **incident** to 1 and 2
- **Degree** = # incident edges
  - E.g. degree of 2 is 4
- **Path** of length **k** from $\mathbf{v_0}$ to $\mathbf{v_k}$
  - Sequence of k+1 nodes <$\mathbf{v_0}$, $\mathbf{v_1}$,…,$\mathbf{v_k}$>, such that every consecutive pair $\mathbf{(v_i,v_{i+1})}$ is an edge in E
  - e.g. <1,2,4,5> is a path of length 3

Undirected graphs (terminology)



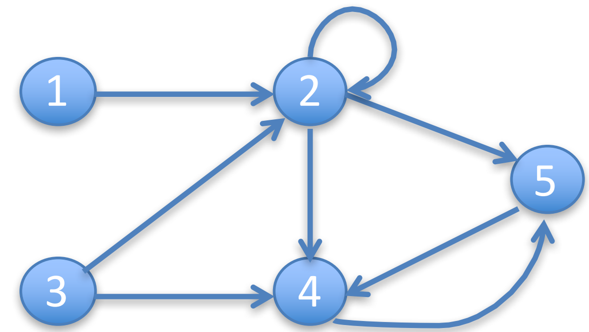
- G(V,E) is **connected** if there is a path between any pair of nodes
- **Cycle** is a path that begins and ends at the same node
  - E.g. <3,2,4,3>
- A graph without cycles is **acyclic**

Directed graphs (terminology)



- Edges are directed, so if (u,v) is an edge it is not necessary that (v,u) is also an edge
- **Outdegree** = # edges leaving a vertex
  - E.g. outdegree of 2 is 3 (Why?)
- **Indegree** = # edges coming into a vertex
  - E.g. indegree of 3 is 0

Directed graphs (terminology)



- **Directed path** of length **k** from $\mathbf{v_0}$ to $\mathbf{v_k}$
  - Note that if there is a path from $\mathbf{v_0}$ to $\mathbf{v_k}$ this does not imply that there is a path from $\mathbf{v_k}$ **to** $\mathbf{v_0}$
- **Directed cycle**
- **Strongly connected digraph:** there us a path between every pair of distinct vertices

Useful properties

- For a undirected graph

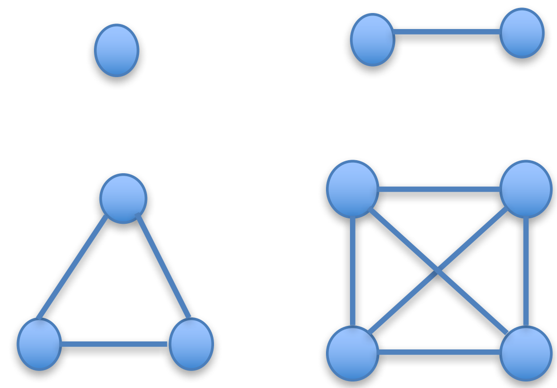
$$\sum_{v \in V} \deg(v) = 2|E|$$

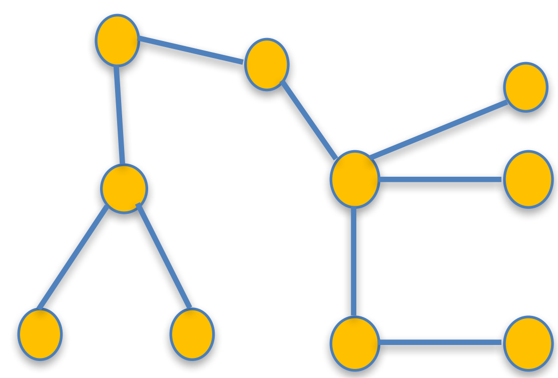
- In a directed graph

$$\sum_{v \in V} in\deg(v) = \sum_{v \in V} out\deg(v) = |E|$$

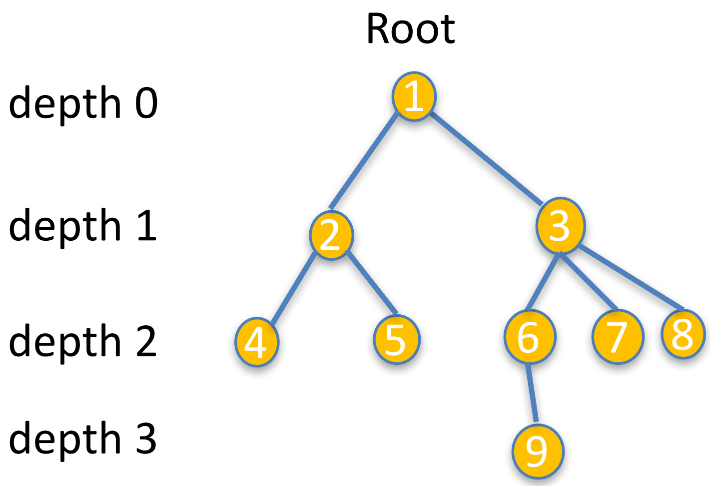
Special Classes of Graphs

- Complete graph (a.k.a. clique)
  - edges between every pair of nodes
  - If |V|=n, then |E|=n(n-1)/2
    - What is the rate of growth of |E|?
- Trees – connected and acyclic
  - Rooted tree (one of the nodes is designated a root)



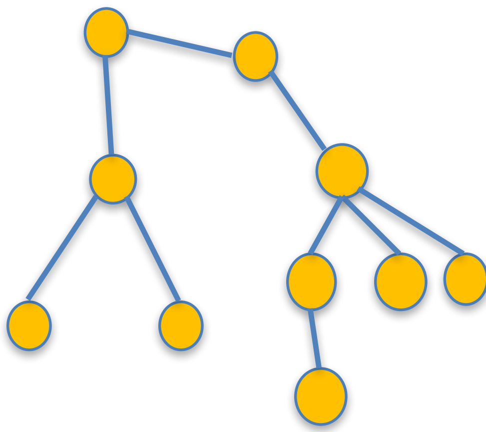


Rooted trees



- 2,3 are <u>**children**</u> of 1
- 8 is a <u>**descendant**</u> of 1
- 3,1 are <u>**ancestors**</u> of 8
- 2 and all its descendants form a <u>**subtree rooted at 2**</u>
- 4,5,9 are <u>**leaves:**</u> nodes with no children
- <u>**Internal nodes:**</u> one or more children (e.g. 3,6)
- <u>**Node height:**</u> path length to the furthest leaf
- <u>**Tree height**</u>: height of the root

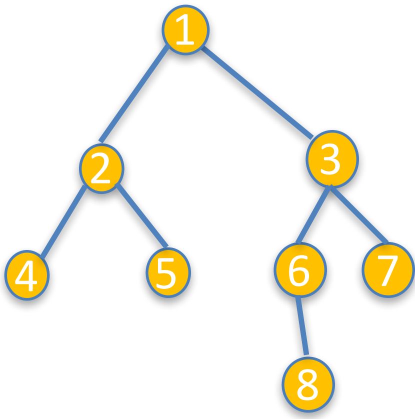
Facts about trees

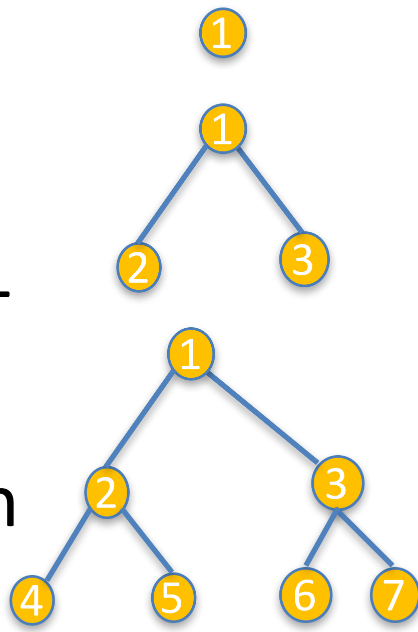


- A unique path connects any pair of nodes
- If $|V|=n$, then $|E|=n-1$
  - What is the rate of growth of $|E|$?
- Deleting any edge disconnects the tree
- Adding any edge results in a cycle

Binary Tree

- Def: each node has at most 2 children
- **Left/Right subtree**:
  - rooted in the left/right child
- **Complete binary tree (CBT)**
  - Each leaf has the same depth
  - Each internal node has two children
- **# internal nodes** in a CBT of height h is 2^h-1
- # leaves in a CBT: 2^h
- Hence height of a CBT with n leaves is $\log_2 n$





Generalization to k-ary tree

- Each node has at most k children
- If complete:
  - No. of internal nodes: $k^n-1/(k-1)$
  - No. of leaves: k^n
  - If we have n leaves height=$\log_k n$

Announcements

- Read through Chapter 3
- HW1 solutions available on BB