403: Algorithms and Data Structures

A Quick Introduction to Graphs and Trees

Fall 2016

UAlbany

Computer Science
Graphs

- A graph $G(V,E)$ consists of
  - Set of nodes (vertices) $V$
  - Set of edges $E$ connecting nodes

- Graphs are extremely useful in modeling problems

- Directed graphs represent binary relations in a set (the nodes). Undirected: symmetric binary relation

(Undirected) graph

Directed graph (Digraph)
Undirected graphs (terminology)

- The edge between 1 and 2 is denoted as (1,2)
  - 1 is **adjacent** to 2
  - (1,2) is **incident** to 1 and 2
- **Degree** = # incident edges
  - E.g. degree of 2 is 4
- **Path** of length k from $v_0$ to $v_k$
  - Sequence of k+1 nodes <$v_0$, $v_1$,...,$v_k$>, such that every consecutive pair ($v_i$, $v_{i+1}$) is an edge in E
  - e.g. <1,2,4,5> is a path of length 3
Undirected graphs (terminology)

- G(V,E) is **connected** if there is a path between any pair of nodes
- **Cycle** is a path that begins and ends at the same node
  - E.g. <3,2,4,3>
- A graph without cycles is **acyclic**
Directed graphs (terminology)

• Edges are directed, so if \((u,v)\) is an edge it is not necessary that \((v,u)\) is also an edge

• **Outdegree** = # edges leaving a vertex
  – E.g. outdegree of 2 is 3 (Why?)

• **Indegree** = # edges coming into a vertex
  – E.g. indegree of 3 is 0
Directed graphs (terminology)

- **Directed path** of length $k$ from $v_0$ to $v_k$
  - Note that if there is a path from $v_0$ to $v_k$ this does not imply that there is a path from $v_k$ to $v_0$

- **Directed cycle**

- **Strongly connected digraph**: there us a path between every pair of distinct vertices
Useful properties

- For a undirected graph
  \[ \sum_{v \in V} \text{deg}(v) = 2 |E| \]

- In a directed graph
  \[ \sum_{v \in V} \text{in\ deg}(v) = \sum_{v \in V} \text{out\ deg}(v) = |E| \]
Special Classes of Graphs

• Complete graph (a.k.a. clique)
  – edges between every pair of nodes
  – If $|V|=n$, then $|E|=n(n-1)/2$
    • What is the rate of growth of $|E|$?

• Trees – connected and acyclic
  – Rooted tree (one of the nodes is designated a root)
Rooted trees

- 2, 3 are children of 1
- 8 is a descendant of 1
- 3, 1 are ancestors of 8
- 2 and all its descendants form a **subtree rooted at 2**
- 4, 5, 9 are **leaves**: nodes with no children
- **Internal nodes**: one or more children (e.g. 3, 6)
- **Node height**: path length to the furthest leaf
- **Tree height**: height of the root
Facts about trees

• A unique path connects any pair of nodes

• If $|V|=n$, then $|E|=n-1$
  – What is the rate of growth of $|E|$?

• Deleting any edge disconnects the tree

• Adding any edge results in a cycle
Binary Tree

• Def: each node has at most 2 children

• **Left/Right subtree:**
  – rooted in the left/right child

• **Complete binary tree (CBT)**
  – Each leaf has the same depth
  – Each internal node has two children

• Number of **internal nodes** in a CBT of height $h$ is $2^h - 1$

• Number of **leaves** in a CBT: $2^h$

• Hence height of a CBT with $n$ leaves is $\log_2 n$
Generalization to k-ary tree

• Each node has at most k children
• If complete:
  – No. of internal nodes: \( k^{n-1}/(k-1) \)
  – No. of leaves: \( k^n \)
  – If we have n leaves height = \( \log_k n \)
Announcements

• Read through Chapter 3
• HW1 solutions available on BB