403: Algorithms and Data Structures

Heapsort and Priority Queues

Fall 2016

UAlbany

Computer Science

Some slides borrowed from David Luebke

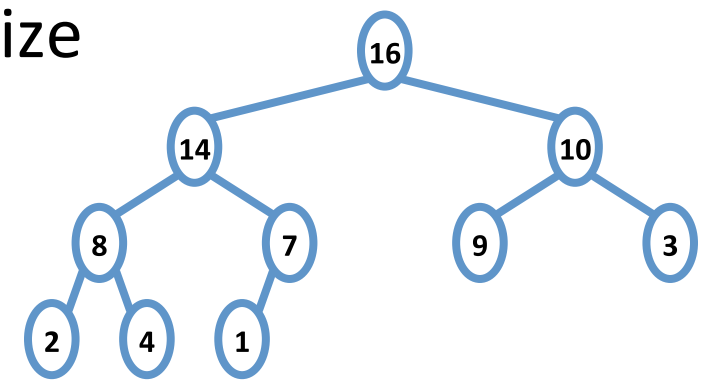
Context

- We defined heaps
  - ”almost” complete binary trees
  - **$A[\mathit{Parent}(i)] \geq A[i]$** for all nodes **$i > 1$**
- Heap operations: Heapify()
  - Fix a single violation of the heap property
  - “Float” values down the tree
  - O(logn), where n is the heap size
    - What is the base of the log?

A =

16	14	10	8	7	9	3	2	4	1

=



Heap Operations: BuildHeap()

- Input: Array **A[1…n]**
- Required: Convert **A** into a heap
- Idea: build a heap in a bottom-up manner by running **`Heapify()`** on successive subarrays

Heap Operations: BuildHeap()

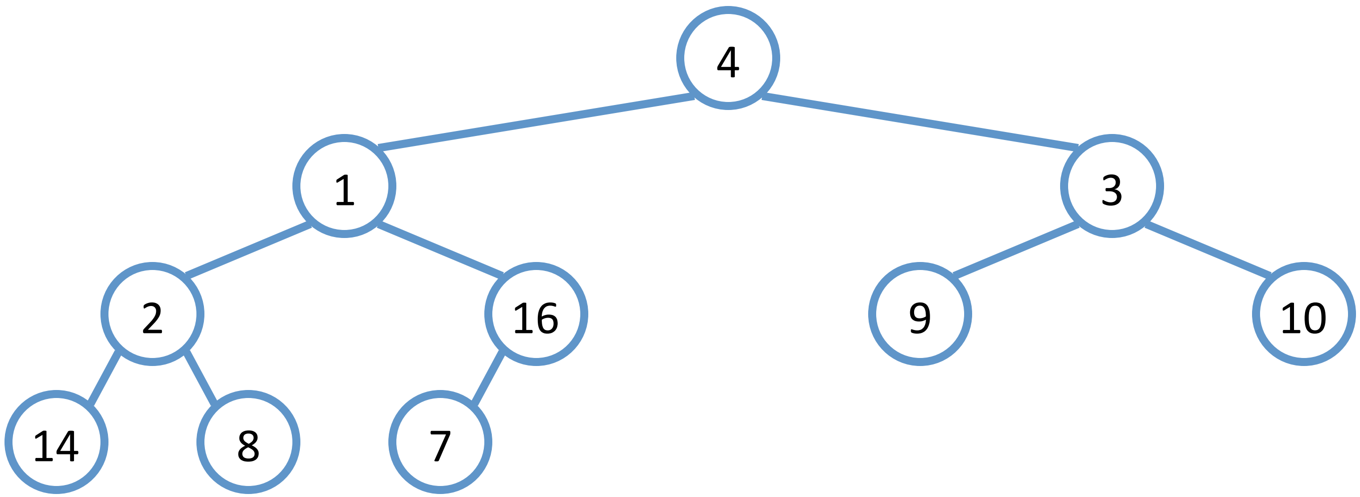
- Fact: for array of length *n*, all elements in range A[$\lfloor n/2 \rfloor$ + 1 .. n] are leaves (*Why?*)
  - Left($\lfloor n/2 \rfloor$ + 1) = 2* ($\lfloor n/2 \rfloor$ + 1) > n
- Another fact: Leaves are (trivially) heaps
- Walk backwards through the array from $\lfloor n/2 \rfloor$ to 1, calling **`Heapify()`** on each node.
  - Order of processing guarantees that the children of node *i* are heaps when *i* is processed
  - Why is this important?

BuildHeap()

```
// given an unsorted array A, make A a heap
BuildHeap(A)
{
  heap_size(A) = length(A);
  for (i = ⌊length[A]/2⌋ downto 1)
     Heapify(A, i);
}
```

BuildHeap() Example

- Work through example
  A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}



Crude Analysis of BuildHeap()

- Each call to **`Heapify()`** takes O(lg n) time
- There are O(n) such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is O(n lg n)
  - *Is this a correct asymptotic upper bound?*
  - *Is this an asymptotically tight bound?*
- A tighter bound is O(n)
  - *How can this be? Is there a flaw in the above reasoning?*

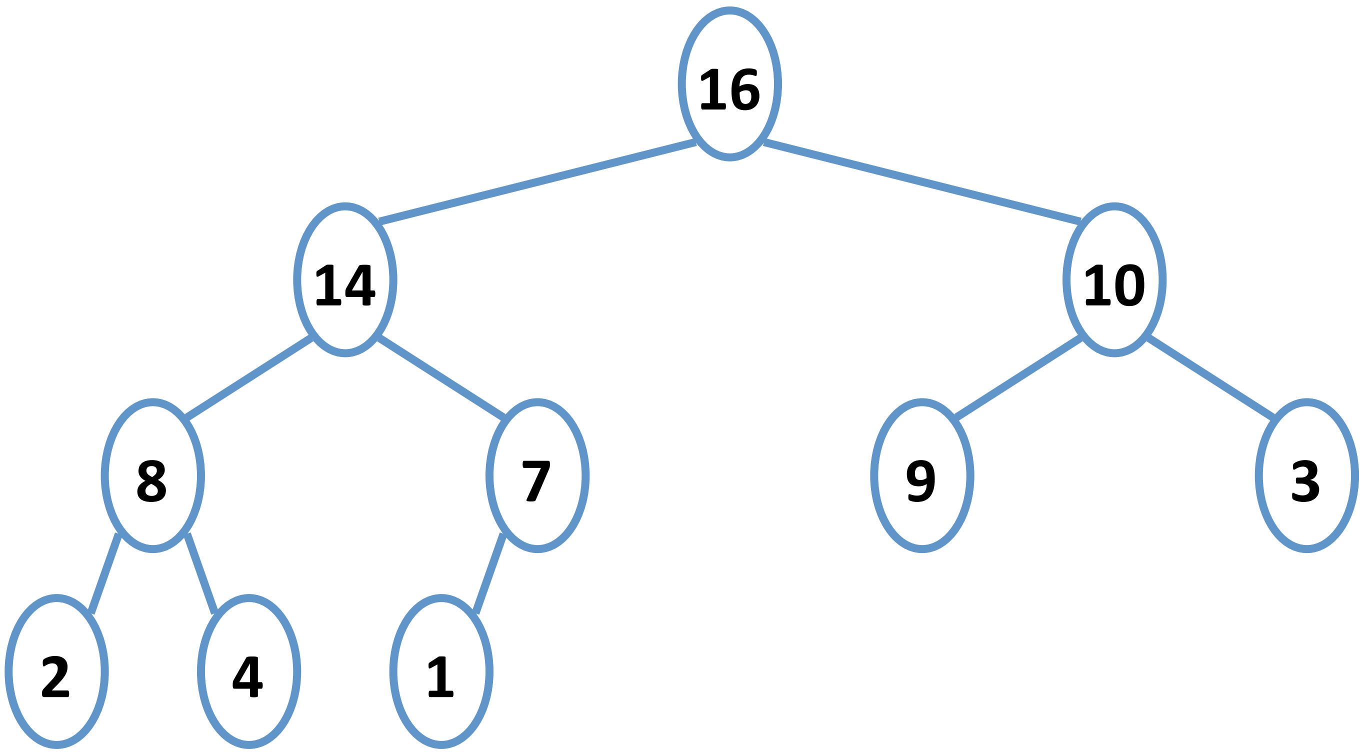
Analyzing BuildHeap(): Tight

- To **`Heapify()`** a subtree takes O(h) time where h is the height of the subtree
  - h = O(lg m), m = # nodes in subtree
  - Intuition: The height of most subtrees is small , i.e. O(log n) is too "generous"
- Fact 1: an n-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h
  - *Proof?*
- Using Fact 1 we can show that **`BuildHeap()`** takes O(n) time
  - Proof?

Heapsort

- Given **`BuildHeap()`**, an in-place sorting algorithm is easily constructed:
  - Maximum element is at A[1]
  - Discard by swapping with element at A[n]
    - Decrement heap_size[A]
    - A[n] now contains correct value
  - Restore heap property at A[1] by calling **`Heapify()`**
  - Repeat, always swapping A[1] for A[heap_size(A)]

Example



Heapsort

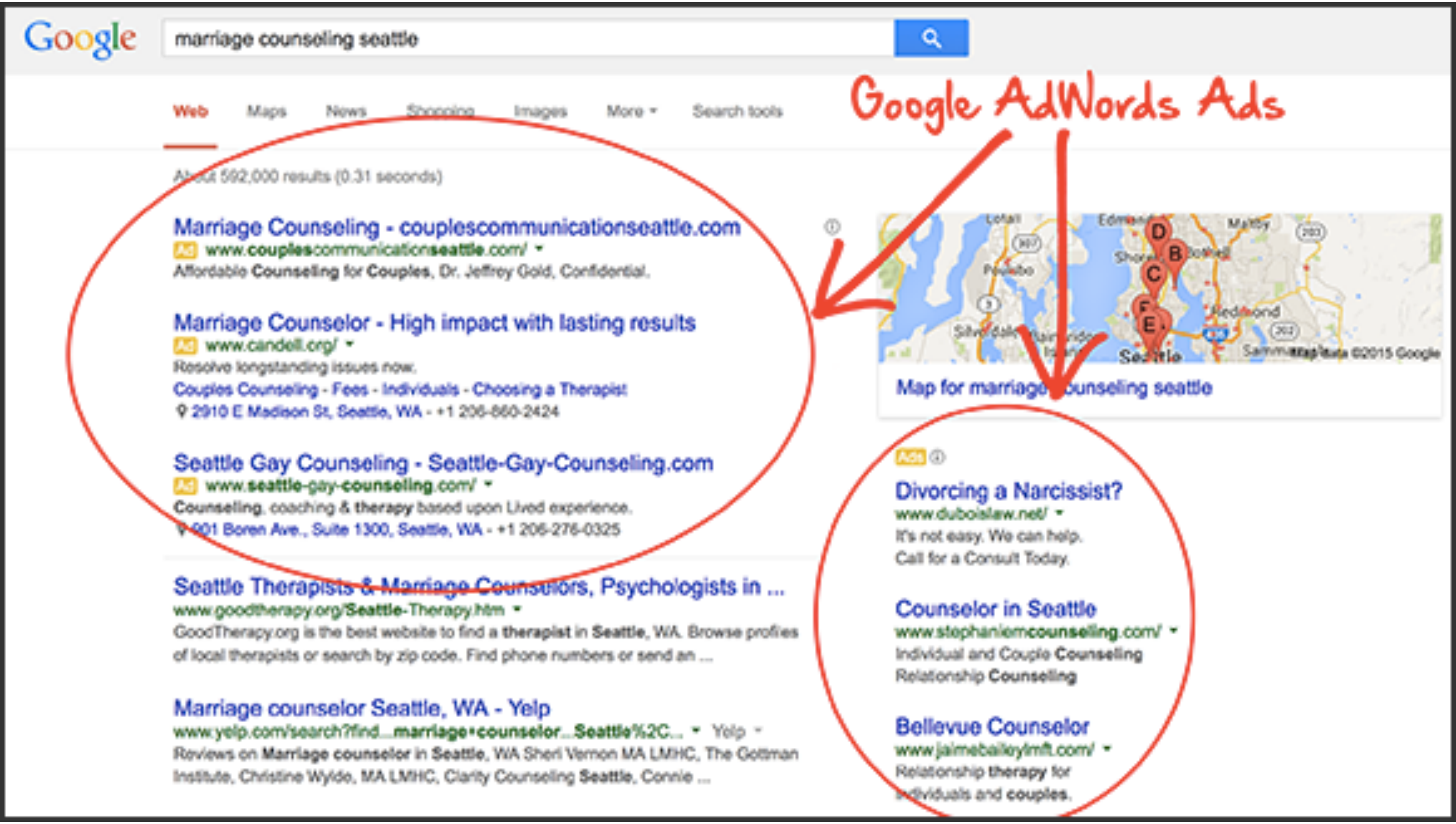
```
Heapsort(A)
{
   BuildHeap(A);
   for (i = length(A) downto 2)
   {
      Swap(A[1], A[i]);
      heap_size(A) -= 1;
      Heapify(A, 1);
   }
}
```

Analyzing Heapsort

- The call to **`BuildHeap()`** takes O(n) time
- Each of the n - 1 calls to **`Heapify()`** takes O(lg n) time
- Thus the total time taken by **`HeapSort()`**
  = O(n) + (n - 1) O(lg n)
  = O(n) + O(n lg n)
  = O(n lg n)

Priority Queues

- Heapsort is a nice algorithm, but in practice Quicksort (coming up) usually wins
- But the heap data structure is incredibly useful for implementing *priority queues*
  - A data structure for maintaining a set *S* of elements, each with an associated value or *key*
  - Supports the operations **`Insert()`**, **`Maximum()`**, and **`ExtractMax()`**
  - ***What might a priority queue be useful for?***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Assassin's prioritized TODO manager

Priority Queue Operations

- **Insert(S, x)** inserts the element x into set S
- **Maximum(S)** returns the element of S with the maximum key
- **ExtractMax(S)** removes and returns the element of S with the maximum key
- *How could we implement these operations using a heap?*

Priority Queue Operations

- **Insert(S, x)**
  - Increment heap size and add x at the end
  - move the new element "upwards" (reverse-heapify)
  - O(log n)
- **Maximum(S)**
  - return S[1]
  - Time complexity?
- **ExtractMax(S)** removes and returns the element of S with the maximum key
  - save S[1], place S[heap_size(S)] in S[1]; Heapify(S,1)
  - Time?

Heap vs All (for Priority queues)

Data Structure	Pre-processing	Insert	Max	Extract Max
Linked List	**O(n)**	**O(1)**	O(n)	O(n)
Sorted Array	O(n logn)	O(n) (shifting)	**O(1)**	**O(1)**
Heap	**O(n)**	O(log n)	**O(1)**	O(log n)

Announcements

- Read through Chapter 6
  - Next class: Build heap, Heap Sort, Priority Queues
- HW2 due next Wednesday