# 403: Algorithms and Data Structures

## Quicksort

Fall 2016

UAlbany

Computer Science

**Some slides borrowed from [David Luebke]()**

# So far: Sorting

| *Algorithm* | *Time* | *Space* |
|---|---|---|
| • **Insertion** | $O(n^2)$ | in-place |
| • **Merge** | O(n logn) | $2^{nd}$ array to merge |
| • **Heapsort** | O(n logn) | in-place |
| • **Quicksort** | from O(n logn) to $O(n^2)$ | in-place |

– very good in practice (small constants)

– Quadratic time is rare

Next

# Quicksort

- Another divide-and-conquer algorithm
  - **DIVIDE:** The array A[p..r] is *partitioned* into two non-empty subarrays A[p..q] and A[q+1..r]
    - Invariant: All elements in A[p..q] are less than all elements in A[q+1..r]
  - **CONQUER:** The subarrays are recursively sorted by calls to quicksort
  - **COMBINE:** Unlike merge sort, no combining step: two subarrays form an already-sorted array

# Quicksort Code

```
Quicksort(A, p, r)
{
    if (p < r)
    {
        q = Partition(A, p, r);
        Quicksort(A, p, q);
        Quicksort(A, q+1, r);
    }
}
```

# Partition

- Clearly, all the action takes place in the **`partition()`** function
  - Rearranges the subarray in place
  - End result:
    - Two subarrays
    - All values in first subarray ≤ all values in second
  - Returns the index of the "pivot" element separating the two subarrays
- *How do you suppose we implement this?*

# Partition In Words

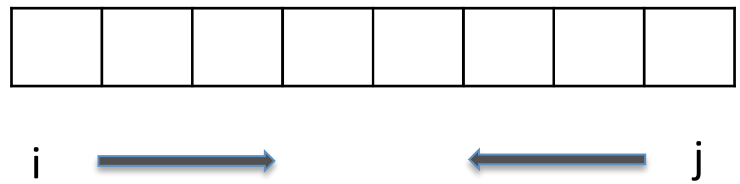
- Partition(A, p, r):
  - Select an element to act as the "pivot" (*which?*)
  - Grow two regions, A[p..i] and A[j..r]
    - All elements in A[p..i] <= pivot
    - All elements in A[j..r] >= pivot
  - Increment i until A[i] >= pivot
  - Decrement j until A[j] <= pivot
  - Swap A[i] and A[j]
  - Repeat until i >= j
  - Return j

**Note: slightly different from book's `partition()`**

# Partition Code

```
Partition(A, p, r)
    x = A[p];
    i = p - 1;
    j = r + 1;
    while (TRUE)
        repeat
            j--;
        until A[j] <= x;
        repeat
            i++;
        until A[i] >= x;
        if (i < j)
            Swap(A, i, j);
        else
            return j;
```

**Choose pivot x**




**Scan looking for element exceeding x**

**Scan looking for element at most x**

**When we find such elements, Exchange them**

```
Illustrate on
A = {4,5,9,7,2,13,6,3};
```

# Example

Pivot=4

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 5 | 9 | 7 | 2 | 13 | 6 | 3 |

i=0 j=9

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 5 | 9 | 7 | 2 | 13 | 6 | 4 |

i=0 i=2 j=5 j=9

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 9 | 7 | 5 | 13 | 6 | 4 |

**j=2** i=3 j=5

**i>j: DONE**

**Goal:**

<=x >=x

Assume all elements are distinct

# Partition Code

```
Partition(A, p, r)
    x = A[p];
    i = p - 1;
    j = r + 1;
    while (TRUE)
        repeat
            j--;
        until A[j] <= x;
        repeat
            i++;
        until A[i] >= x;
        if (i < j)
            Swap(A, i, j);
        else
            return j;
```

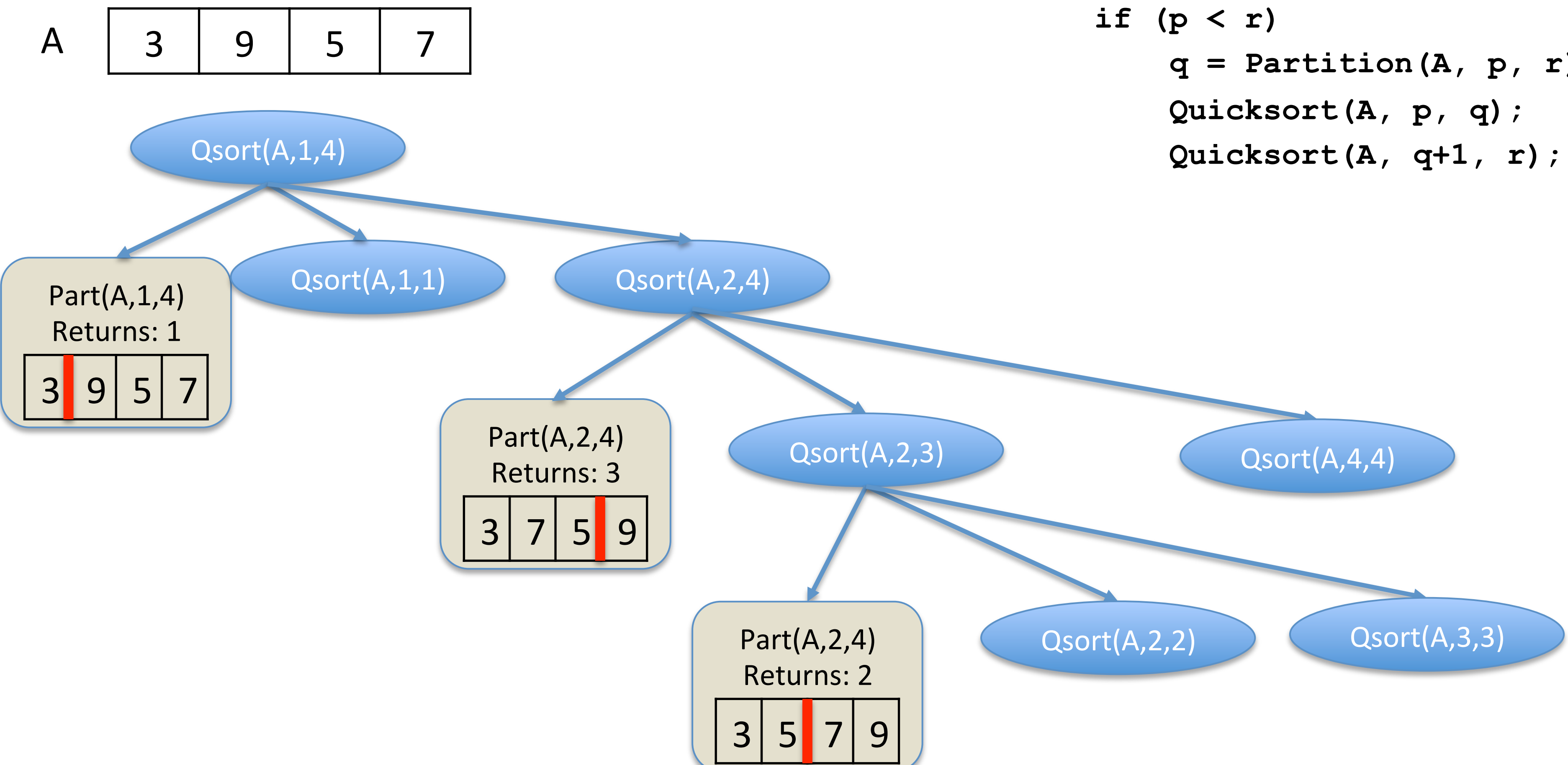
What is the running time of `partition()`?

`partition()` runs in O(n) time

- O(1) at each element: skip or swap
- Linear in the size of the array

# Back to Quicksort

```
Quicksort(A, p, r)
    if (p < r)
        q = Partition(A, p, r);
        Quicksort(A, p, q);
        Quicksort(A, q+1, r);
```



# Analyzing Quicksort

- *What will be a bad case for the algorithm?*
  – Partition is always unbalanced
- *What will be the best case for the algorithm?*
  – Partition is perfectly balanced
- *Which is more likely?*
  – The latter, by far, except...
- *Will any particular input elicit the worst case?*
  – Yes: Already-sorted input

# Analyzing Quicksort: Balanced splits

- In the balanced split case:

  $T(n) = 2T(n/2) + \Theta(n)$

- What does this work out to?

  $T(n) = \Theta(n \lg n)$

**Take home:** **A good balance is important**

# Analyzing Quicksort: Sorted case

- Sorted case:

  $T(1) = \Theta(1)$

  $T(n) = T(n - 1) + \Theta(n)$

  by substitution…

  $T(n) = T(1) + n\Theta(n)$

- Works out to

  $T(n) = \Theta(n^2)$

| 2 | 3 | 6 | 7 | 10 | 13 | 14 | 16 |
|---|---|---|---|---|---|---|---|

First call: j will decrease to 1 (n steps)
Second: j decrease to 2 (n-1 steps)
…
$n + n-1 + n-2 + \ldots = \Theta(n^2)$

# Is sorted really the worst case?

- Argue formally that things cannot get worse
- A formal argument with gen
- Assume that every split resu
  - Size q
  - Size n-q
- $T(n) = \max_{1<=q<=n-1}[T(q)+T(n-$
  - where $T(1) = O(1)$
- <u>Show that $T(n) = O(n^2)$</u>

# Average behavior: Intuition

- Worst case: assumes 1:n-1 split
  - rare in practice
- The O(nlogn) behavior occurs even if the split is say 10%:90%
- If all splits are equally likely
  - 1:n-1, 2:n-2 … n-1:1
  - then on average, we will not get a very tall tree
  - details in extra slide at the end (not required)

# Avoiding the $O(n^2)$ case

- The real liability of quicksort is that it runs in $O(n^2)$ on already-sorted input
- Solutions
  - Randomize the input array
  - *Pick a random pivot element*
  - choose 3 elements and take median for pivot
- *How will these solve the problem?*
  - By ensuring that no particular input can be chosen to make quicksort run in $O(n^2)$ time

# Other Improvements (lower constants)

- When a subarray is small (say smaller than 5) switch to a simple sorting procedure say insertion sort instead of Quicksort
  - why does this help?
- Pick more than one pivot
  - Partitions the array in more than 2 parts
  - Smaller number of comparisons (1.9nlogn vs 2nlogn ) and overall better performance in practice
  - Details: Kushagra et al. “Multi-Pivot Quicksort: Theory and Experiments”, SIAM, 2013

# Announcements

- Read through Chapter 7
- HW2 due on Wednesday

# Extra slides*

- Average case rigorous analysis follows
- This is advanced material (will not appear in HWs and exam)

# Analyzing Quicksort: Average Case

- Assuming random input, average-case running time is much closer to O(n lg n) than O($n^2$)
- First, a more intuitive explanation/example:
  - Suppose that partition() always produces a 9-to-1 split. This looks quite unbalanced!
  - The recurrence is thus:

    $T(n) = T(9n/10) + T(n/10) + n$

    **Use n instead of O(n) for convenience (how?)**
  - *How deep will the recursion go?*

# Analyzing Quicksort: Average Case

- Intuitively, a real-life run of quicksort will produce a mix of "bad" and "good" splits
  - Randomly distributed among the recursion tree
  - Pretend for intuition that they alternate between best-case (n/2 : n/2) and worst-case (n-1 : 1)
  - *What happens if we bad-split root node, then good-split the resulting size (n-1) node?*

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  - *What happens if we bad-split root node, then good-split the resulting size (n-1) node?*
    - We fail English

# Analyzing Quicksort: Average Case

- Intuitively, a real-life run of quicksort will produce a mix of "bad" and "good" splits
  - Randomly distributed among the recursion tree
  - Pretend for intuition that they alternate between best-case (n/2 : n/2) and worst-case (n-1 : 1)
  - *What happens if we bad-split root node, then good-split the resulting size (n-1) node?*
    - We end up with three subarrays, size 1, (n-1)/2, (n-1)/2
    - Combined cost of splits = n + n -1 = 2n -1 = O(n)
    - No worse than if we had good-split the root node!

# Analyzing Quicksort: Average Case

- Intuitively, the O(n) cost of a bad split (or 2 or 3 bad splits) can be absorbed into the O(n) cost of each good split
- Thus running time of alternating bad and good splits is still O(n lg n), with slightly higher constants
- How can we be more rigorous?

# Analyzing Quicksort: Average Case

- For simplicity, assume:
  - All inputs distinct (no repeats)
  - Slightly different **`partition()`** procedure
    - partition around a random element, which is not included in subarrays
    - all splits (0:n-1, 1:n-2, 2:n-3, … , n-1:0) equally likely
- *What is the probability of a particular split happening?*
- Answer: 1/n

# Analyzing Quicksort: Average Case

- So partition generates splits
  (0:n-1,  1:n-2,  2:n-3, … ,  n-2:1,  n-1:0)
  each with probability 1/n
- If T(n) is the expected running time,

$$T(n) = \frac{1}{n}\sum_{k=0}^{n-1}\left[T(k) + T(n-1-k)\right] + \Theta(n)$$

- *What is each term under the summation for?*
- *What is the $\Theta(n)$ term for?*

# Analyzing Quicksort: Average Case

- So…

$$T(n) = \frac{1}{n} \sum_{k=0}^{n-1} \left[ T(k) + T(n-1-k) \right] + \Theta(n)$$

$$= \frac{2}{n} \sum_{k=0}^{n-1} T(k) + \Theta(n)$$

**Write it on the board**

  - Note: this is just like the book's recurrence (p166), except that the summation starts with k=0
  - We'll take care of that in a second

# Analyzing Quicksort: Average Case

- We can solve this recurrence using the dreaded substitution method
  - Guess the answer
  - Assume that the inductive hypothesis holds
  - Substitute it in for some value < n
  - Prove that it follows for n

# Analyzing Quicksort: Average Case

- We can solve this recurrence using the dreaded substitution method
  - Guess the answer
    - *What's the answer?*
  - Assume that the inductive hypothesis holds
  - Substitute it in for some value < n
  - Prove that it follows for n

# Analyzing Quicksort: Average Case

- We can solve this recurrence using the dreaded substitution method
  - Guess the answer
    - $T(n) = O(n \lg n)$
  - Assume that the inductive hypothesis holds
  - Substitute it in for some value < n
  - Prove that it follows for n

# Analyzing Quicksort: Average Case

- We can solve this recurrence using the dreaded substitution method
  - Guess the answer
    - $T(n) = O(n \lg n)$
  - Assume that the inductive hypothesis holds
    - *What's the inductive hypothesis?*
  - Substitute it in for some value < n
  - Prove that it follows for n

# Analyzing Quicksort: Average Case

- We can solve this recurrence using the dreaded substitution method
  - Guess the answer
    - $T(n) = O(n \lg n)$
  - Assume that the inductive hypothesis holds
    - $T(n) \le an \lg n + b$ for some constants $a$ and $b$
  - Substitute it in for some value < n
  - Prove that it follows for n

# Analyzing Quicksort: Average Case

- We can solve this recurrence using the dreaded substitution method
  - Guess the answer
    - $T(n) = O(n \lg n)$
  - Assume that the inductive hypothesis holds
    - $T(n) \le an \lg n + b$ for some constants $a$ and $b$
  - Substitute it in for some value < n
    - *What value?*
  - Prove that it follows for n

# Analyzing Quicksort: Average Case

- We can solve this recurrence using the dreaded substitution method
  - Guess the answer
    - $T(n) = O(n \lg n)$
  - Assume that the inductive hypothesis holds
    - $T(n) \le an \lg n + b$ for some constants $a$ and $b$
  - Substitute it in for some value < n
    - The value $k$ in the recurrence
  - Prove that it follows for n

# Analyzing Quicksort: Average Case

- We can solve this recurrence using the dreaded substitution method
  - Guess the answer
    - $T(n) = O(n \lg n)$
  - Assume that the inductive hypothesis holds
    - $T(n) \le an \lg n + b$ for some constants $a$ and $b$
  - Substitute it in for some value < n
    - The value $k$ in the recurrence
  - Prove that it follows for n
    - Grind through it…

# Analyzing Quicksort: Average Case

$$T(n) = \frac{2}{n}\sum_{k=0}^{n-1} T(k) + \Theta(n)$$

**The recurrence to be solved**

$$\le \frac{2}{n}\sum_{k=0}^{n-1} (ak\lg k + b) + \Theta(n)$$

**Plug in inductive hypothesis**

$$\le \frac{2}{n}\left[b + \sum_{k=1}^{n-1} (ak\lg k + b)\right] + \Theta(n)$$

**Expand out the k=0 case**

$$= \frac{2}{n}\sum_{k=1}^{n-1} (ak\lg k + b) + \frac{2b}{n} + \Theta(n)$$

**2b/n is just a constant, so fold it into Θ(n)**

$$= \frac{2}{n}\sum_{k=1}^{n-1} (ak\lg k + b) + \Theta(n)$$

**Note: leaving the same recurrence as the book**

# Analyzing Quicksort: Average Case

$$T(n) = \frac{2}{n}\sum_{k=1}^{n-1}(ak\lg k + b) + \Theta(n)$$

**The recurrence to be solved**

$$= \frac{2}{n}\sum_{k=1}^{n-1} ak\lg k + \frac{2}{n}\sum_{k=1}^{n-1} b + \Theta(n)$$

**Distribute the summation**

$$= \frac{2a}{n}\sum_{k=1}^{n-1} k\lg k + \frac{2b}{n}(n-1) + \Theta(n)$$

**Evaluate the summation: b+b+…+b = b (n-1)**

$$\leq \frac{2a}{n}\boxed{\sum_{k=1}^{n-1} k\lg k} + 2b + \Theta(n)$$

**Since n-1<n, 2b(n-1)/n < 2b**

This summation gets its own set of slides later

# Analyzing Quicksort: Average Case

$$T(n) \le \frac{2a}{n}\sum_{k=1}^{n-1} k\lg k + 2b + \Theta(n)$$

**The recurrence to be solved**

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + 2b + \Theta(n)$$

**We'll prove this later**

$$= an\lg n - \frac{a}{4}n + 2b + \Theta(n)$$

**Distribute the (2a/n) term**

$$= an\lg n + b + \left(\Theta(n) + b - \frac{a}{4}n\right)$$

**Remember, our goal is to get T(n) ≤ an lg n + b**

$$\le an\lg n + b$$

**Pick a large enough that an/4 dominates Θ(n)+b**

# Analyzing Quicksort: Average Case

- So $T(n) \leq an \lg n + b$ for certain $a$ and $b$
  - Thus the induction holds
  - Thus $T(n) = O(n \lg n)$
  - Thus quicksort runs in $O(n \lg n)$ time on average (phew!)
- Oh yeah, the summation…

# Tightly Bounding
# The Key Summation

$$\sum_{k=1}^{n-1} k\lg k = \sum_{k=1}^{\lceil n/2\rceil -1} k\lg k + \sum_{k=\lceil n/2\rceil}^{n-1} k\lg k$$

**Split the summation for a tighter bound**

$$\leq \sum_{k=1}^{\lceil n/2\rceil -1} k\lg k + \sum_{k=\lceil n/2\rceil}^{n-1} k\lg n$$

**The lg k in the second term is bounded by lg n**

$$= \sum_{k=1}^{\lceil n/2\rceil -1} k\lg k + \lg n\sum_{k=\lceil n/2\rceil}^{n-1} k$$

**Move the lg n outside the summation**

# Tightly Bounding
# The Key Summation

$$\sum_{k=1}^{n-1} k \lg k \le \sum_{k=1}^{\lceil n/2 \rceil -1} k \lg k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

**The summation bound so far**

$$\le \sum_{k=1}^{\lceil n/2 \rceil -1} k \lg(n/2) + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

**The lg k in the first term is bounded by lg n/2**

$$= \sum_{k=1}^{\lceil n/2 \rceil -1} k(\lg n - 1) + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

**lg n/2 = lg n - 1**

$$= (\lg n - 1) \sum_{k=1}^{\lceil n/2 \rceil -1} k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

**Move (lg n - 1) outside the summation**

# Tightly Bounding
# The Key Summation

$$\sum_{k=1}^{n-1} k \lg k \le (\lg n - 1) \sum_{k=1}^{\lceil n/2 \rceil - 1} k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

**The summation bound so far**

$$= \lg n \sum_{k=1}^{\lceil n/2 \rceil - 1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

**Distribute the (lg n - 1)**

$$= \lg n \sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k$$

**The summations overlap in range; combine them**

$$= \lg n \left( \frac{(n-1)(n)}{2} \right) - \sum_{k=1}^{\lceil n/2 \rceil - 1} k$$

**The Guassian series**

# Tightly Bounding
# The Key Summation

$$\sum_{k=1}^{n-1} k \lg k \le \left(\frac{(n-1)(n)}{2}\right)\lg n - \sum_{k=1}^{\lceil n/2 \rceil - 1} k$$

**The summation bound so far**

$$\le \frac{1}{2}\left[n(n-1)\right]\lg n - \sum_{k=1}^{n/2-1} k$$

**Rearrange first term, place upper bound on second**

$$\le \frac{1}{2}\left[n(n-1)\right]\lg n - \frac{1}{2}\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)$$

**X Guassian series**

$$\le \frac{1}{2}\left(n^2 \lg n - n \lg n\right) - \frac{1}{8}n^2 + \frac{n}{4}$$

**Multiply it all out**

# Tightly Bounding
# The Key Summation

$$\sum_{k=1}^{n-1} k \lg k \le \frac{1}{2}\left(n^2 \lg n - n \lg n\right) - \frac{1}{8} n^2 + \frac{n}{4}$$

$$\le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \text{ when } n \ge 2$$

Done!!!