Scaling is one of several kinds of useful **transformations**. EG. $x' = 3x$, $y' = y$, $z' = z$ expands things and distances in the $\pm x$ direction away from the $x = 0$ (yz plane) by a multiplying them by 3.

Here is the matrix form for our body-thickening transformation:

$$\left[\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 0y + 0z \\ 0x + 1y + 0z \\ 0x + 0y + 1z \end{bmatrix} = \begin{bmatrix} 3x \\ y \\ z \end{bmatrix}$$

is a **transformation** that transforms vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ into } \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 3x \\ y \\ z \end{bmatrix}$$

Our scaling transformation, in matrix form:

$$\begin{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

transforms

$$\text{unit vector} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{into} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

and the other two unit vectors into *themselves*.

Transformations given by a matrix M **acting on** a vector V by matrix multiplication $M\ V$ are linear transformations.

Such transformations are called **linear** because they satisfy, first, this proposition:

$$M\ (\alpha \mathbf{V}) = (\alpha)\ M\ \mathbf{V}$$

(examine the matrix multiplication formula to see this.. Remember scalar multiplication $\alpha(v_1, v_2, v_3) = (\alpha v_1, \alpha v_2, \alpha v_3)$)

And second, another proposition:

$$M\ (\mathbf{V} + \mathbf{W}) = M\ \mathbf{V} + M\ \mathbf{W}$$

(examine the multiplication formula again...it's the result of basic number facts like $m_{ij}(v_j + w_j) = m_{ij}v_j + m_{ij}w_j$ and that vector addition is done componentwise.)

Geometric significance of linearity:

$(M\ (\alpha\mathbf{V}) = (\alpha)\ M\ \mathbf{V})$ If all the vectors in our picture of scalar multiplication are transformed by the same linear transformation, the new picture will illustrate correct scalar mutltiplication (with the same scalar).

$(M\ (\mathbf{V}+\mathbf{W}) = M\ \mathbf{V} + M\ \mathbf{W})$ Similarly, if a parallelogram or tail-to-head picture of vector addition were transformed linearly, the new picture will be of correct vector addition.

Linearity of M in a nutshell (one popular formula):

$$M\ (\alpha\mathbf{V} + \beta\mathbf{W}) = \alpha M\ \mathbf{V} + \beta M\ \mathbf{W}$$

Also, the formulas are **identities**: Always true, no matter what values the variables have.

Our variables: α, β, $\mathbf{V}$ and $\mathbf{W}$.

Unit coordinate vectors:

$$\mathbf{U}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{U}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{U}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

An *arbitrary vector*'s components play the following role in writing is using unit vectors:

$$\mathbf{V} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x\mathbf{U}_1 + y\mathbf{U}_2 + z\mathbf{U}_3.$$

1. A matrix-vector multiplication and a linear transformation are essentially the same things geometrically but the multiplication is most practical for computers to compute.

2. A linear transformation M IS DETERMINED BY what it does to each of the unit coordinate vectors and indeed, by what it does to a "basis" Why??

$$M\,\mathbf{V} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= M\,(x\mathbf{U}_1 + y\mathbf{U}_2 + z\mathbf{U}_3)$$

$$= x\,M\,\mathbf{U}_1 + y\,M\,\mathbf{U}_2 + z\,M\,\mathbf{U}_3$$

$$= x\,M \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\,M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\,M \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So.. How do you figure out what matrix to use for a linear transformation?

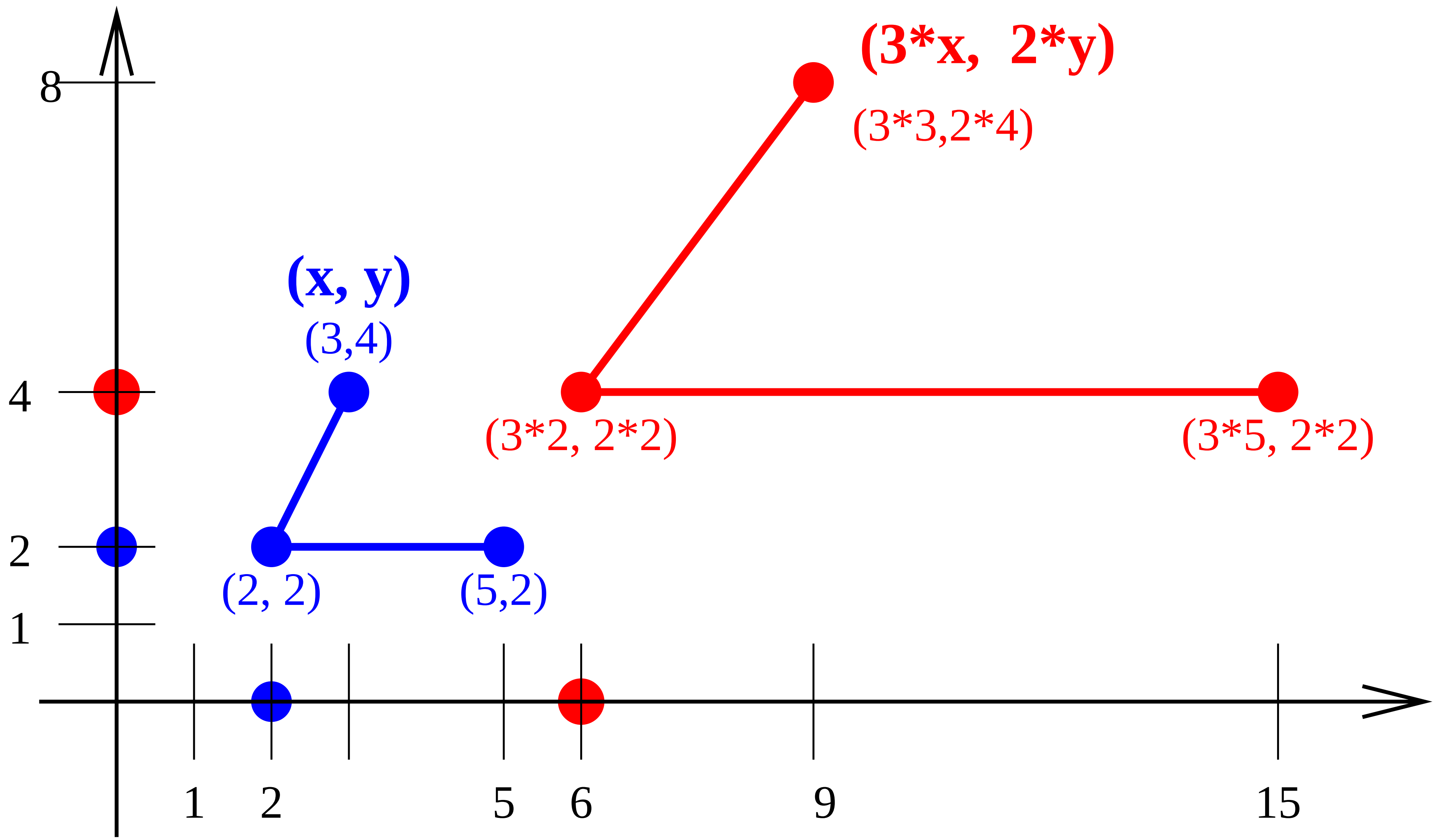
Answer: Figure out what 3 vectors each of the 3 coordinate unit vectors should transform into, and write them for the columns of the matrix to use.

(And take care to maintain consistancy of names and order..)

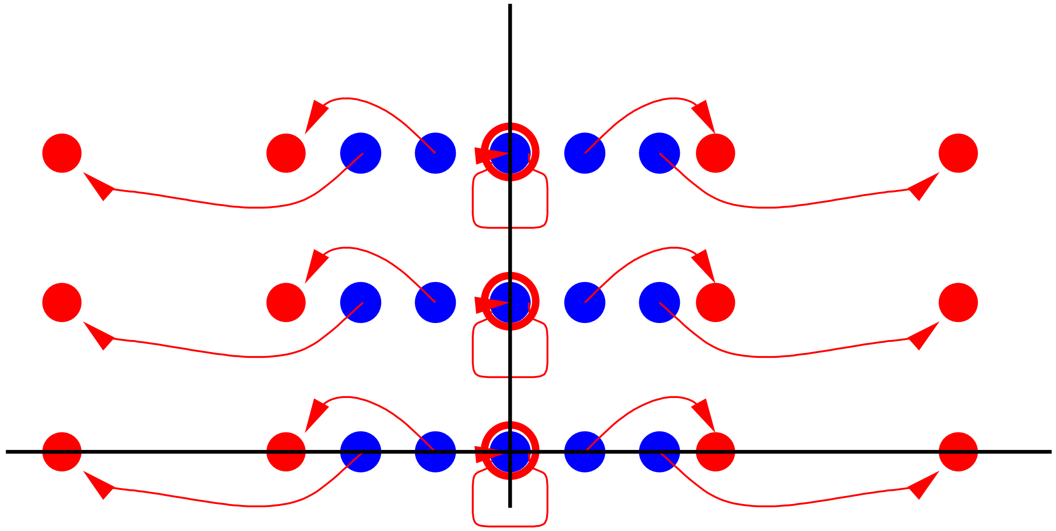
scaling: $x' = (S_x)x$, $y' = (S_y)y$, $z' = (S_z)z$

Stretches the space by a (different) factor in each axis direction, with the origin fixed.

Angles and shapes change, except horizontal, vertical and front-to-back lines remain unchanged.



Where the object is relative to the origin is important for scaling.

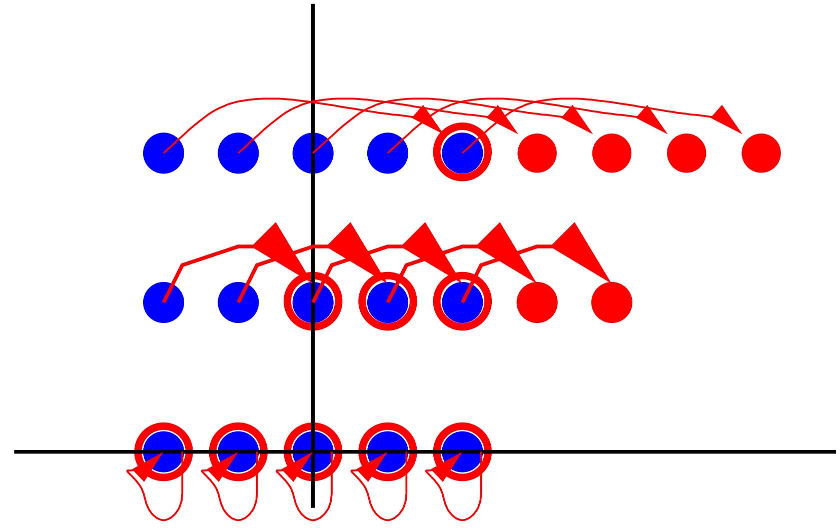


Scaling

x'=3x

y'=1y

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$



Shearing

x'=x+y

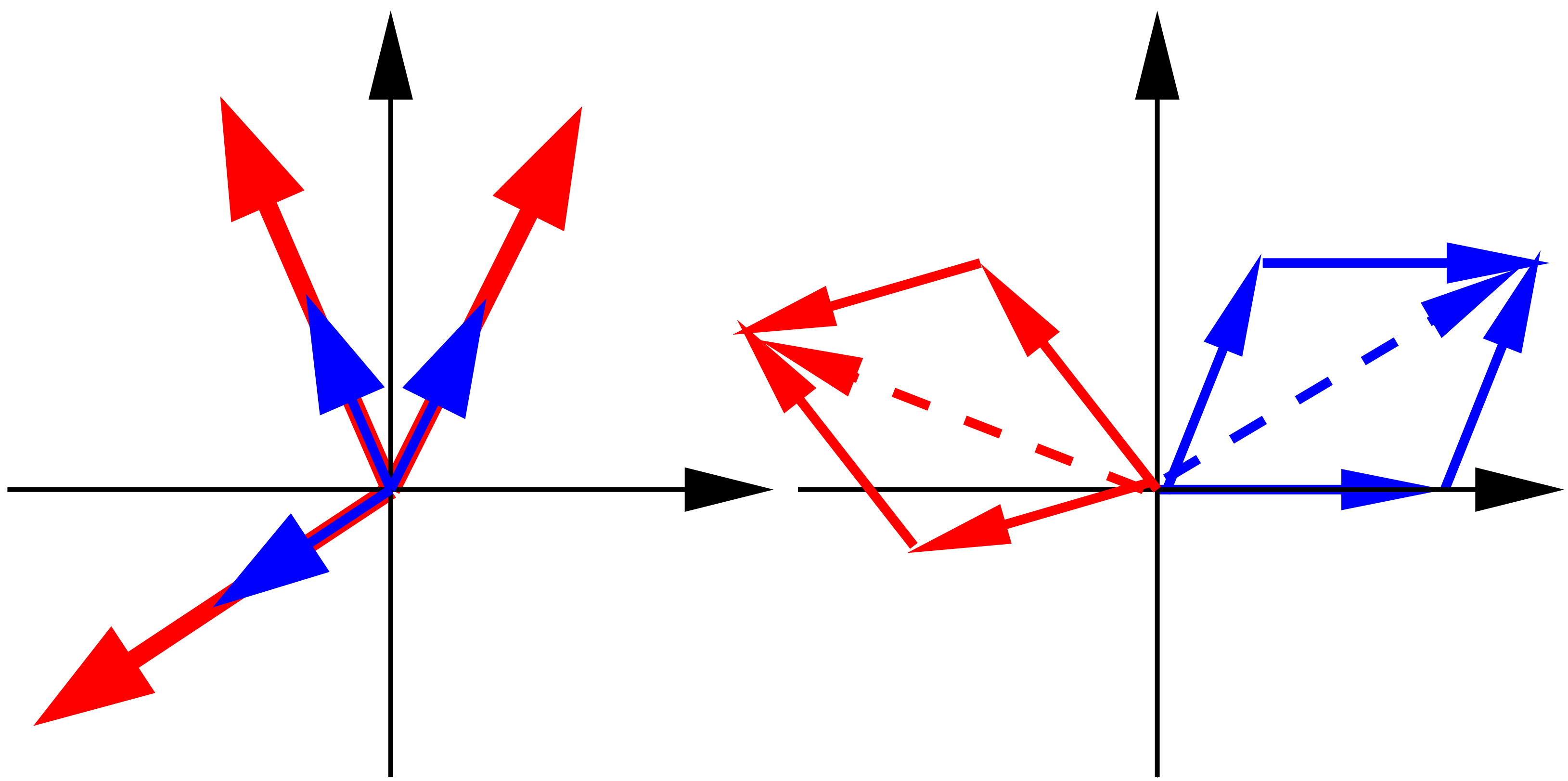
y'=y

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

For shearing: $M \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $M \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Geometric Fact: Rotation around the z-axis is a **rigid motion** that leaves the origin unchanged. So, it is a *linear transformation*.

Directly, scalar multiplication and the parallelogram (tail-to-head) figures remain valid for fixed-origin rotation.



So, to figure the matrix that *rotates space by angle* Θ *around the* z *axis:*

1. Figure out, using sin, cos, and the 3 axes, the coordinates of the vectors obtained by rotating each unit vector.
2. Arrange the resulting three coordinate vectors as the columns of the matrix.

Matrix multiplication:

$$\begin{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} & \begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \end{bmatrix} & \begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} m_{11}x_1 + m_{12}x_2 + m_{13}x_3 \\ m_{21}x_1 + m_{22}x_2 + m_{23}x_3 \\ m_{31}x_1 + m_{32}x_2 + m_{33}x_3 \end{bmatrix}$$

is a **transformation** that transforms vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ into } \begin{bmatrix} m_{11}x_1 + m_{12}x_2 + m_{13}x_3 \\ m_{21}x_1 + m_{22}x_2 + m_{23}x_3 \\ m_{31}x_1 + m_{32}x_2 + m_{33}x_3 \end{bmatrix}$$

What is the fate of the three **unit vectors**

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

under this transformation?

Matrix

$$\begin{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} & \begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \end{bmatrix} & \begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \end{bmatrix} \end{bmatrix}$$

transforms

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \end{bmatrix}$ which is **column 2**

and it transforms

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ into **column 3**.

Vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is transformed into

$$x \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} + y \begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \end{bmatrix} + z \begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \end{bmatrix}$$

In other words, vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

is transformed into

$$\begin{bmatrix} m_{11}x \\ m_{21}x \\ m_{31}x \end{bmatrix} + \begin{bmatrix} m_{12}y \\ m_{22}y \\ m_{32}y \end{bmatrix} + \begin{bmatrix} m_{13}z \\ m_{23}z \\ m_{33}z \end{bmatrix} = \begin{bmatrix} m_{11}x + m_{12}y + m_{13}z \\ m_{21}x + m_{22}y + m_{23}z \\ m_{31}x + m_{32}y + m_{33}z \end{bmatrix}$$