1. A matrix-vector multiplication and a linear transformation are essentially the same things geometrically but the multiplication is most practical for computers to compute.

2. A linear transformation M IS DETERMINED BY what it does to each of the unit coordinate vectors and indeed, by what it does to a "basis" Why??

$$M\ \mathbf{V} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= M\ (x\mathbf{U}_1 + y\mathbf{U}_2 + z\mathbf{U}_3)$$

$$= x\ M\ \mathbf{U}_1 + y\ M\ \mathbf{U}_2 + z\ M\ \mathbf{U}_3$$

$$= x\ M \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\ M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\ M \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So.. How do you figure out what matrix to use for a linear transformation?

Answer: Figure out what 3 vectors each of the 3 coordinate unit vectors should transform into, and write them for the columns of the matrix to use.

(And take care to maintain consistancy of names and order..)

Apply this to derive the 2-d rotation matrix R_Θ:

The rotation by angle Θ transforms

$$U_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ into } R_\Theta(U_x) = \begin{bmatrix} \cos(\Theta) \\ \sin(\Theta) \end{bmatrix}$$

and

$$U_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ into } R_\Theta(U_y) = \begin{bmatrix} -\sin(\Theta) \\ \cos(\Theta) \end{bmatrix}$$

So the matrix is

$$R_\Theta = \begin{bmatrix} \cos(\Theta) & -\sin(\Theta) \\ \sin(\Theta) & \cos(\Theta) \end{bmatrix}$$

Picture of $\sin(\Theta)$, $\cos(\Theta)$, and coordinates of rotated U_x and U_y under construction.

The x, y, z coordinates of *any* unit vector $\mathbf{W}$ are

- The projections of $\mathbf{W}$ on the x, y, z axes $\mathbf{U}_x$, $\mathbf{U}_y$, $\mathbf{U}_z$.
- The **dot** products $\mathbf{W} \cdot \mathbf{U}_x$, $\mathbf{W} \cdot \mathbf{U}_y$, $\mathbf{W} \cdot \mathbf{U}_z$.
- The 3 *cosines* of the 3 angles that $\mathbf{W}$ makes with $\mathbf{U}_x$, $\mathbf{U}_y$, $\mathbf{U}_z$.

So these 3 numbers describe the *direction* in which $\mathbf{W}$ points. They are called **direction cosines**.

Translation is simple but apparently non-linear.

$$T_{(t_x,t_y)}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Here, it is expressed using vector addition to add a vector containing the **parameters** t_x and t_y.

It cannot be expressed by a 2×2 matrix multiplication because it is non-linear.

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) \neq T(\mathbf{0}) + T(\mathbf{0}) = \begin{bmatrix} 2t_x \\ 2t_y \end{bmatrix}$$

But, graphics experts use **homogeneous coordinates**. At first, this looks like just an easy to learn trick. We'll see its important geometric significance for **projections** later.

Idea: Use 3-component vectors for 2-dim planes, 4-component vectors for 3-dim space.

When the extra component is 1, a matrix multiplication represents translation:

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$

Combination of a rotation FIRST, followed by a TRANSLATION, is expressed by

$$\begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & t_x \\ \sin(\Theta) & \cos(\Theta) & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\Theta)x - \sin(\Theta)y + t_x \\ \sin(\Theta)x + \cos(\Theta)y + t_y \\ 1 \end{bmatrix}$$

This combination is the matrix product:

$$\begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & t_x \\ \sin(\Theta) & \cos(\Theta) & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But the same 2 matrices multiplied THE OTHER WAY give a different product matrix:

$$\begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & \cos(\Theta)t_x - \sin(\Theta)t_y \\ \sin(\Theta) & \cos(\Theta) & \sin(\Theta)t_x + \cos(\Theta)t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Compare to

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & t_x \\ \sin(\Theta) & \cos(\Theta) & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

The translation displacement is ROTATED when the translation is done first.