Anticommutative Tutte Functions and Unimodular Oriented Matroids

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Abstract

The Tutte equations are ported (or set-pointed) when the equations $F(N) = g_e F(N/e) + r_e F(N\backslash e)$ are omitted for elements $e$ in a distinguished set called ports. Their solutions, called ported Tutte functions, can distinguish different orientations of the same matroid. A ported extensor with ground set is a decomposable element (antisymmetric tensor) in the exterior algebra over a vector space with a designated ground set basis containing a distinguished subset called ports. We give exterior algebra operations that correspond to oriented matroid deletion, contraction and dualization, and use them to define a ported extensor function. We then prove that this function obeys a sign corrected exterior algebraic variant of the ported Tutte equations. For unimodular extendors without ports, this function reduces to the basis enumerator; and then, for graphs, to the Laplacian (or Kirchhoff) determinant. On graphs with port edges, the function value, an extensor, signifies the space of solutions to Kirchhoff’s and Ohm’s electricity equations after projection to the voltage and current variables associated to the ports. Indeed, the Laplacian matrix with the identity matrix appended is a case of the construction we present.

Tree and forest enumeration expressions for electrical resistance are generalized. We also demonstrate how the corank-nullity polynomial, basis expansions with activities, and a geometric lattice expansion generalize to ported Tutte functions of oriented matroids. The ported Tutte functions are parametrized, which raises the problem of how to generalize known characterizations of parameterized non-ported Tutte functions.

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1 Introduction

A ported unoriented or oriented matroid $\mathcal{N} = \mathcal{N}(P, E)$ has its ground set $S(\mathcal{N}) = P \cup E$ given with a distinguished subset $P$ of elements which we call ports; $P \cap E = \emptyset$. The following definition extends the parametrized Tutte equations and functions studied by Zaslavsky [57] and Bollobas and Riordan [5] with results unified by Ellis-Monaghan and Traldi [21]. Let two parameters $g_e$ and $r_e$ be given for each $e \in E$.

Definition 1.1. A function $F$ is a ported Tutte function if the domain of $F$ is a minor closed class of ported unoriented or oriented matroids and $F$ satisfies the following ported Tutte Equations for each $\mathcal{N}$ in the class:

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When $e \in E$ is a non-separating element, i.e., $e$ is neither a port nor loop nor a coloop (i.e., isthmus):

$$F(\mathcal{N}) = g_e F(\mathcal{N}/e) + r_e F(\mathcal{N} \setminus e).$$  

(1)

When $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ with ground sets $S(\mathcal{N}_1) \cap S(\mathcal{N}_2) = \emptyset$:

$$F(\mathcal{N}_1 \oplus \mathcal{N}_2) = F(\mathcal{N}_1)F(\mathcal{N}_2).$$  

(2)

1.1 Ports and Parameters

The idea to restrict deletion/contraction decomposition so it does not apply to some distinguished elements is a natural one. It was applied to invariants of matroid morphisms (or strong maps) by Las Vergnas [22, 31, 32]. The equations for ported Tutte invariants (i.e., $F$ with the $r_e = g_e = 1$ and invariant under port preserving matroid isomorphisms) determine the **big Tutte polynomial** defined by Las Vergnas in [32]. Whereas the traditional Tutte polynomial has only two variables, which correspond to the isomorphism classes of the indecomposable loop and coloop matroids, the generalization adds additional variables corresponding to the additional indecomposable matroids on sets of port elements. We had applied this idea to study the behavior of such polynomials under a restricted matroid union operation [11], which is one generalization of matroid series connection. Our motivation there and here comes from distinguishing as ports some edges in a directed graph model of an electrical network. The ports model two-terminal plugs or sockets where the network interacts with an enviroment; the other edges model resistors.

In this paper we add the observation that if ported Tutte decomposition is done on an oriented matroid, the indecomposable minors include connected oriented matroids on sets of port elements. Therefore, if $\mathcal{N}_1$ and $\mathcal{N}_2$ are different orientations of the same orientable matroid, then $F(\mathcal{N}_1) \neq F(\mathcal{N}_2)$ is possible for a ported Tutte function $F$. That is the reason we mention that $\mathcal{N}$ is unoriented or oriented in the definition of a ported Tutte function. The big Tutte polynomial, like traditional Tutte functions, does not distinguish different orientations of the same matroid.

The parameters are naturally motivated by the electrical resistance of each non-port edge. Two, $g_e$ (for conductance) and $r_e$ (for resistance) are used (as in [39]) so that all resistances including zero and infinity can be modelled uniformly. Parametrized Tutte functions of graphs including forest enumerators have other applications [40]. For more applications and results arising especially from parametrization, see [5, 21, 57] and the references therein. Our results are more clearly expressed and no harder to prove than if the parameters were omitted. Furthermore, the characterizations of parametrized Tutte functions in [5, 21, 57] raise the natural questions of generalization with ports (see §5). Fortunately, our current results do not entail the complicating problems that arise when parameters are added to the Tutte equations.

In our application, port elements never have parameters because there is no electrical resistance directly associated with an edge used to demark a pair of vertex terminals. Deletion and contraction of the non-port edges corresponds, for us, to eliminating non-port voltage and current variables. The result is projection of solutions to the variables associated to the port elements.

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1.2 Laplacians and Electrical Networks

We further motivate the subject by some ideas about how we will generalize the basis enumerator

\[
F_B(\mathcal{N}) = \sum_{B \in B(\mathcal{N})} g_B r_B.
\]

(3)

In the case of a graph with all the \( r_e = 1 \), this is the reduced Laplacian (or Kirchhoff) determinant which enumerates the weighted spanning trees. The discrete Laplacian matrix plays an important role in the theory of electrical networks and their many analogs. The following observations underlie the combination of ideas that we will present: First, the basis enumerator is a Tutte function that happens to be a determinant in the case of unimodular (or regular) matroids, such as the graphic matroids. Second, topics involving determinants, including the chirotope presentation of realizable oriented matroids, can profitably be studied with exterior algebra. (See below and §2.) Third, non-trivial electrical networks (see §6.2) must have port elements for supplying power or for connecting to an external environment, in addition to the resistor elements usually modelled by graph edges. We are interested in their combinatorial properties beyond spanning tree counts [8–14]. Finally, electrical flows and potential differences are inherently directional. The patterns of their directions (expressed by sign functions on graph edges) feasible under Kirchhoff’s two laws are precisely the vector and covector families, respectively, of the graphic oriented matroid. Indeed, the duality between these laws, current and voltage, is characterized by oriented matroid theory.

For one class of matroids, the number of bases is easy to compute by a determinant. A unimodular (or regular) matroid \( \mathcal{N} \) (by [55, see Theorem 3.1.1]) is presented by the column dependencies of a full row rank locally unimodular matrix \( N \). Let \( D = \text{diag}(g_e/r_e) \), where the \( e \in S \) index the columns of \( N \). Then

\[
r_S \det(NDN^d) = r_S \sum_{X \subseteq S} g_X N[X]^2 = r_S \sum_{B \in B(\mathcal{N})} \frac{g_B}{r_B} = \sum_{B \in B(\mathcal{N})} g_B r_B = F_B(\mathcal{N})
\]

(4)

by the Cauchy-Binet theorem and the locally unimodular property that every minor \( N[X] \) indexed by columns \( X \) satisfies \( N[X] = \pm 1 \) or 0 depending on whether \( X \) is a basis in \( \mathcal{N} \). (\( NDN^d \) is the reduced Laplacian matrix if all \( r_e = 1 \) and \( N \) is the reduced oriented incidence matrix of a graph with edge weights \( g_e \).) The function \( r_S \det(NDN^d) \) can be applied to an arbitrary full row rank matrix \( N \) to obtain an easy-to-compute basis generating function, but the coefficients, although positive, will not all be 1 when \( N \) is not locally unimodular. There is evidence suggesting that computing the number of bases is intractible for non-unimodular matroids [48].

When the port set is empty, our main construction (Definition 3.1) reduces to the above determinant. When \( P \) is identified with the vertex set the function value is basically the whole Laplacian matrix—see §6.2. It is well-known and easy to verify that the spanning tree count is a Tutte invariant of graphs. The fact that the reduced Laplacian determinant enumerates the spanning trees is the famous Matrix Tree Theorem. Our results further elucidate the place of this theorem in the enumeration methods for resistive electrical
network solutions pioneered by Kirchhoff [30] and Maxwell [36]. These methods were introduced into combinatorial theory by Brooks, Smith, Stone and Tutte [7, 45] who attributed them to Kirchhoff, see §6.1. They continue to be applied within some electrical engineering computer aided design tools [23]. Two points of departure from [7] are to replace analysis in terms of graph vertices and incidences by analysis of functions on the graph edges, and then to express the relevant equations (§6.2) in exterior algebra.

1.3 Some Graph Tutte Functions in Exterior Algebra

The exterior algebra is generated from the vectors in a finite dimensional linear space $K S$ by the free antisymmetric (multilinear) product. We call the decomposable elements extensors. The non-zero extensors, up to scalar multiples, correspond to linear subspaces. The distinguished basis $S$ determines a basis for the exterior algebra. This basis is essentially all the subsets of $S$, each written in a particular order. The Plücker coordinates of an extensor are the coefficients when the extensor is expressed with this basis. The Plücker coordinates, a vector of determinants, are the maximal minors of a matrix whose rows are a basis for the subspace which the extensor corresponds to. See §2.1 and §2.2, and the survey by White [56].

Exterior algebra enables us to present unoriented and oriented matroids by extensors where the basis $S$ is specified as the ground set. In addition, the matroid operations in the Tutte equations can be mimicked by exterior algebra operations. Our main constructions (Definitions 3.1 and 4.3) give functions that obey variants (Theorems 3.3 and 4.5) of the ported Tutte equations in which the (commutative) ring operations are replaced by exterior algebra sum and product. Since the Plücker coordinates are the determinants of maximal submatrices in a full-rank matrix representation of an unoriented or oriented matroid, the matroid bases and the oriented matroid chirotope can be determined from the signs of the Plücker coordinates. In fact, identities among extensors such as Theorem 3.3 are equivalent to systems of identities among determinants.

We will now introduce our main results in cases of graphs with one or more port edges. Theorems 3.3 and 4.5 generalize the fact that spanning tree count, a case of the determinant (4), is a Tutte function. The function value can be expressed with a particular basis; the coefficients are the value’s Plücker coordinates (see Definition 2.2). We forewarn the reader that a non-empty set of ports is necessary for the generalization to be new—where the number of Plücker coordinates is more than one.

The first interesting case is illustrated by a graph with one port edge, say edge $p$. It is the simplest (non-zero rank) case of an extensor-valued Tutte function. The extensor expresses, in a homogeneous form, the equivalent resistance of a two-terminal network. For simplicity, let all the parameters be 1. The Kirchhoff enumeration methods include the following surprising yet classical result, one of the “easily remembered” rules stated without proof by Maxwell [35] (see section 6.3). Let all edges $\neq p$ represent unit resistors in an electrical network. The purpose of the port edge $p$ is to demark two terminal vertices. Maxwell’s rule asserts that the equivalent resistance between the two terminal vertices equals the quotient of the count $A$ of spanning trees that contain edge $p$ divided by the
count $B$ of spanning trees that omit edge $p$. Note that whenever a two-tree spanning forest $F$ for which $F \cup p$ is a tree counted by $A$ is contracted and the other non-$p$ edges deleted, $p$ becomes a coloop. When this is done for $B$, $p$ becomes a loop. $A$ and $B$ are the two Plücker coordinates of the extensor we construct in this paper. They are the coefficients in the linear equation $A_i + B(-v_p) = 0$ that relates the port current and the (negated) port voltage. The extensor (Definition 3.1) will generalize this expression of the electrical network behavior observable at one port to any finite number of ports. The reader can verify that each of $A$ and $B$ satisfy the additive ported Tutte equation; so $(A, B)$ satisfies it. (It is instructive to calculate how $(A, B)$ behaves under the multiplicative ported Tutte equation.) When the number of ports is $q$, we generalize $(A, B)$ to a $(2q)$-tuple of Plücker coordinates. The illustrated electrical significance of the Plücker coordinate ratios, and the enumerative significance of each one, will be generalized for all $q$.

In some cases of graphs with two or more port edges, some of these Plücker coordinates will equal the difference between the counts of two kinds of spanning forests. Such a coordinate pertains to the influence of a fixed voltage or a fixed current constraint at one port upon a quantity observed at a different port. Our results show how oriented matroid properties determine the sign by which each forest $F$ contributes to this coordinate (Corollary 3.8). In particular, that sign is determined by the graphic oriented matroid on the (directed) port edges obtained by contracting $F$ and deleting the remaining non-port edges. The simplest case where distinct signs do occur is when both orientations of the same 2-circuit matroid on two ports appear in this process. The contribution of $F$ to the Plücker coordinate with a given index $X$ is calculated in a particularly simple way: We solve the electrical network of port edges only (with no resistors!) that resulted from this deletion and contraction, after checking matroid rank conditions necessary for at least one coordinate to be non-zero. The contribution is the Plücker coordinate with the same index $X$ from the solution of the latter, port-edge-only network. It should be noted that the sign of the contribution is determined by the oriented graphic matroid on ports only, independent of the particular $F$ contracted to obtain this oriented matroid. Details appear in our proof of Maxwell's rule for two ports, section 6.3.

It is amusing go back to the one port $p$ case and derive the Maxwell's rule coefficients $(A, B)$ from the principles we just illustrated. The port behavior of the network consisting of one coloop $p$ is defined by $1 \cdot i_p + 0 \cdot v_p = 0$: This constraint can be expressed by Plücker coordinates $(1, 0)$. The port current is 0 and the port voltage is unconstrained since the graph has no cycles. Dually, the port behavior of the one port loop network is expressed by $(0, 1)$. The port voltage is 0 but the current is unconstrained. One consequence of our main theorem proves that $A(1, 0) + B(0, 1) = (A, B)$ gives the Plücker coordinates of the constraint that the original graph imposes on its port current and (negated) port voltage variables, in accordance with Maxwell's rule $A_i + B(-v_p) = 0$. Our theory justifies the choices of $(1, 0)$ and $(0, 1)$ rather than any other multiple $(\alpha, 0)$ or $(0, \beta)$ in $A(\alpha, 0) + B(0, \beta)$. 
1.4 General Extensors and Identities

The determinants and extensors we study are not just those involved in graphs and electrical networks. The first step to address the above electrical network analysis and forest enumeration problems was to express a graphic oriented matroid by an extensor. The function in Definition 3.1 maps this extensor to an extensor that represents the solution to the problems. However, the definition is meaningful on any extensor over the linear space $K^S$, as long as a particular basis $S$ and a distinguished port subset of $S$ are given. In general, for any extensors over a vector space with a distinguished basis (to serve as the ground set), we can define deletion, contraction, duality and direct sum operations that represent the corresponding unoriented or oriented matroid operations on the corresponding matroids.

Our main result (Theorem 3.3) is that the extensor function we construct satisfies a variant of the ported Tutte equations, where all the matroid and (commutative) ring operations are replaced by the analogous operations on extensors. Since exterior multiplication is anticommutative (see (5) in §2.2), our variant requires sign correction factors that depend on the order in which the ground set elements are given in the construction of the function.

Our extensor forms of oriented matroid operations and our sign correction factors are expressed using notation where ground set elements have arbitrary rather than integer names. Instead of permutation sign, we utilize a sign function $\epsilon$ on sequences of ground set elements. This ground set orientation is arbitrary except for being alternating and non-zero on sequences of distinct elements. It facilitates calculations when several sequences with different underlying sets occur in one formula. It is a combinatorial adaptation of orientations of manifolds and definitions of pseudo-forms (de Rham’s “forms of odd-kind”), see §6.4.

Unimodular extensors are those where all the Plücker coordinates are 0 or ±1. Each unimodular extensor corresponds uniquely up to sign to a unimodular (or regular) oriented matroid. We find that our function, restricted to unimodular extensors, determines a ported Tutte function on ported unimodular oriented matroids (Theorem 4.5). The values of this function are extensors.

Like other Tutte functions, this function has a Tutte polynomial expression. First, for all oriented matroids (not just unimodular) we can define the ported corank-nullity polynomial. It is like Las Vergnas’ big Tutte polynomial except the matroid variables are oriented and it has parameters. When the given oriented matroid is unimodular, each oriented matroid monomial (being unimodular since it is a minor of the original) corresponds to a unique extensor, up to sign. This extensor is the function value on an indecomposable minor on some port elements. Our theory specifies a sign choice for each minor. When those properly signed extensors are substituted (and the assignments of $u = 0$ and $v = 0$ used classically to enumerate bases are made), the result is the extensor value of our Tutte function (Theorem 4.4).

The extensor function we construct is computable using linear algebra in ways similar to the Laplacian determinant (see §3). This determinant has a unique status in Tutte
function and invariant computation. Evidence reviewed in §6.5 indicates little hope for other interesting determinant based or other easy-to-compute Tutte functions or invariants under non-ported Tutte equations and matroid isomorphism. This is another motivation for extending the Tutte function theory to ported matroids: The computation of matrix expressions (see [16, §12.4]) equivalent to our extensor function are routine linear algebra exercises used in electrical network analysis.

We now proceed to the details about exterior algebra and Plücker coordinates pertaining to realizable non-oriented and oriented matroids. They include algebraic operations and identities which correspond to some elementary matroid relationships. Our main construction, an extensor valued function of ported extentors, and the ported Tutte equation variant that it satisfies, is presented in §3. A variant of the corank-nullity polynomial in §4 is used to express our function restricted to unimodular extentors. The variant differs from Las Vergnas’ big Tutte polynomial so (1) it applies to oriented matroids instead of unoriented matroids, and (2) it includes parameters as in Definition 1.1. Extensions to ported matroids of known results about expressing Tutte functions as set, basis and flat expansions, and the related open questions follow in §5. Further discussion of context is given in §6 and brief remarks on peripheral topics appear in §7.

2 Preliminaries

Throughout, $K$ denotes a field, either the reals, rationals, or their extensions generated by the parameters $g_e, r_e$.

2.1 Exterior Algebra

We refer the reader to basic texts such as [28, §7.1-7.2, on associative and exterior algebras over fields] for complete development and proofs. The following is a synopsis with the emphasis on the facts we will need. It also explains certain notational conventions which help to mimic oriented matroid theory in exterior algebra. We use a combinatorial approach to adapt the operations of Hodge star (for duality) and tensor contraction (for matroid contraction).

An associative algebra $\mathcal{A}$ over $K$ is a ring that is also a vector space over $K$, for which addition and 0 are the same in both the ring and the vector space, and for which the ring and scalar multiplications are compatible: $a(xy) = (ax)y = x(ay)$ where $a \in K$ and $x, y \in \mathcal{A}$.

Let $V$ be the vector space $KS$ where finite set $S$ is a basis. Thus $V$ consists of the all $\sum_{e \in S} a_e e$, $a_e \in K$, where $\sum a_e e = 0$ if and only if $a_e = 0$ for every $e \in S$. The associative algebra over $K$ generated by $V$ consists of all finite $K$-linear combinations of 1 (the ring identity) and formal finite products of elements of $S$.

The exterior algebra $\mathcal{E}(V)$ over $V$ is the quotient of the associative algebra over $K$ generated by $V$ modulo the algebra ideal $I$ generated by products $v^2, v \in V$. The image of each $v \in V$ under the map $v \rightarrow v + I \in \mathcal{E}(V)$ is denoted by $\mathbf{v}$. These $\mathbf{v}$ will also be called vectors. Thus, for $v_1, v_2 \in V$, $(\mathbf{v}_1 + \mathbf{v}_2)(\mathbf{v}_1 + \mathbf{v}_2) = 0$, $\mathbf{v}_1 \mathbf{v}_1 = 0$ and so $\mathbf{v}_1 \mathbf{v}_2 = -\mathbf{v}_2 \mathbf{v}_1$.
in $\mathcal{E}(V)$. This and the associativity law imply that each product of a sequence of vectors vanishes if the sequence has repeated elements. Indeed the product vanishes if and only if there is a linear dependency among the vectors being multiplied. Note that a non-zero product of two or more vectors is not a vector.

A particular basis of $\mathcal{E}(V)$ is constructed from the basis $S = \{s_1, \ldots, s_n\}$ of $V$. This basis consists of $1$ together with the $2^n - 1$ products of vectors from distinct non-empty subsets of $\{s_1, \ldots, s_n\}$, each product written in a particular order. A formula for exterior product expressed in terms of this basis is used in [28] to prove that the product is associative. The formula expresses the following fact for products of basis vectors which is true for all products: Given any sequence of vectors $v_1 \cdots v_k$ and permutation $\sigma \in \mathfrak{S}_k$ with sign $\epsilon(\sigma)$, the exterior product satisfies the alternating law

$$v_1 \cdots v_k = \epsilon(\sigma)v_{\sigma_1} \cdots v_{\sigma_k}.$$ 

As a result, $\mathcal{E}(V)$ is an associative algebra that has dimension $2^n$ when viewed as a vector space over $K$. It is customary to use increasing order of subscripts, so each $X \in \mathcal{E}(V)$ can be expressed by

$$X = x_0 1 + \sum_{\emptyset \neq A \subseteq S;\ A = \{s_{i_1}, \ldots, s_{i_k}\},\ i_1 < \cdots < i_k} x_A s_{i_1} \cdots s_{i_k}$$

with $2^n$ unique coefficients $x_A, A \subseteq S$.

We follow a different convention which mimics the one used with the chirotope cryptomorphism for oriented matroids given in [4]. For us, $A$ will denote an arbitrary sequence of elements of $S$. The value in $K$ symbolized by coefficient $x_A$ will depend on the order as well as the elements of $A$, but these values will satisfy

$$x_{A_{\sigma}} = \epsilon(\sigma)x_A$$

where $A_{\sigma} = (a_1 \cdots a_k)_{\sigma} = a_{\sigma_1} \cdots a_{\sigma_k}$ is $A$ permuted by $\sigma$. In general, a function is called alternating if it has this property. Our convention allows $A$ to have repeated elements but requires $x_A = 0$ for such $A$.

We follow a related convention for subset expansions and formulas within them. When necessary, a set symbol like $A$ in $A \subseteq S$ will denote distinct elements written in an arbitrary sequence. But the expansion or formula will be written only if its value is independent of the sequence chosen for each symbol. Furthermore, when $A$ is a sequence of distinct basis vectors, the corresponding product of their images in $\mathcal{E}(V)$ will be denoted by $A$. The empty sequence $\emptyset$ corresponds to $1 \in \mathcal{E}(V)$. No sequence corresponds to $0 \in \mathcal{E}(V)$. The above $2^n$ term basis expansion is thus written

$$X = \sum_{A \subseteq S} x_A a_1 \cdots a_{|A|} = \sum_{A \subseteq S} x_A A.$$

This expansion follows our convention because $x_{A_{\sigma}} A_{\sigma} = \epsilon^2(\sigma)x_A A = x_A A$. Note that $X = 0$ if and only if every $x_A = 0$. 

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2.2 Extensors with Ground Set

The exterior algebra $\mathcal{E}(V)$ is a powerful tool to explore, in a coordinate-free way, relationships between linear subspaces of $V$ [2, 56]. These relationships are, in other words, the theorems about the projective geometry whose points are the rank 1 (zero-dimensional) subspaces. The geometric flats (the empty set, points, lines, planes, etc., and the whole space) correspond to these linear subspaces. One way to present a $K$-realizable matroid is to map each ground set element to either a point or to the empty flat in this projective geometry. The matroid structure is then expressed in terms of incidences of the images of the ground set elements with the geometric flats. The expression is coordinate-free because these relationships do not change under a change of the basis $S$ for $V$. We mention this presentation method to contrast it with our application of exterior algebra.

In our application, each $K$-realizable matroid with ground set $S$ will be presented by a separate (decomposable) element $N \in \mathcal{E}(V)$, where $V = KS$. Each such element $N$ will determine a linear subspace $L = L(N)$ of $KS$. Consider the family of matrices with columns indexed by $S$ whose row space equals $L$. The matroid is presented by the linear dependencies among the columns of any such matrix. When $L$ is viewed as a linear subspace of functions from $S$ to $K$, the matroid is the “function space geometry (or chain-group geometry) $G(S, L)$” discussed in [54, §1.1.C]. As such, each $e \in S$ corresponds to the linear functional given by evaluation of $f \in L$ on $e$. Therefore these functionals, as a finite subset of the dual space of $L$, comprise a vector representation of the matroid.

The members of $L$ present an oriented matroid $\mathcal{N}(N)$ by a set of covectors $\mathcal{L}$. Each covector is the function $l : S \to \{+,-,0\}$ determined by some $f \in L$ by $l(e) = \text{sign}(f(e))$ for all $e \in S$. The signed cocircuits are the covectors with minimal non-empty support. See [1] for an exposition of oriented matroids that begins with linear subspace presentations including the cycle and coboundary (or cocycle) spaces of graphs. Our topic emphasizes exterior algebra and the chirotope given by the signs of the Plücker coordinates of $N$ to present oriented matroids. Theorem 2.1 states the needed details. Deeper discussions appear in [4, especially §2.4 on stratifications of the Grassmann variety and chap. 8 on realizations].

The following theorem summarizes the facts we will need. It characterizes those elements of $\mathcal{E}(V)$ (as decomposable) that determine linear subspaces and present the $K$-realizable matroids with ground set $S$. We will call such elements extensors.

**Theorem 2.1.** Given non-zero element $N \in \mathcal{E}(V)$ where $V = KS$, the following conditions are equivalent:

1. There exist $r$ linearly independent vectors $v_i \in V$ for which

   $$N = v_1 \cdots v_r.$$

   (When $r = 0$, the condition is $N = \alpha 1$ for some $\alpha \in K, \alpha \neq 0$.)

   $L$ is the subspace spanned by $\{v_1, \ldots, v_r\}$. 
2. There exists \( r, 0 \leq r \leq |S| \), such that the only non-zero coefficients \( N[A] \) in
\[
N = \sum_{A \subseteq S} N[A] A
\]
satisfy \( |A| = r \), and the function \( N[A] \) from sequences \( A \) to \( K \) is alternating and satisfies the Grassmann-Plücker relationships:
For all length \( r \) sequences \( A = a_1 \cdots a_r \) and \( B = b_1 \cdots b_r \) over \( S \),
\[
N[A]N[B] = \sum_{i=1}^{r} N[a_1 a_2 \cdots a_i b_i \cdots a_r]N[b_1 \cdots a_i a_1 \cdots a_r].
\]
(Here, \( a_i b_i \) means \( b_i \) within sequence \( B \) is replaced by \( a_i \).)

3. There exists a rank \( r \) matrix \( N \) with \( r \) rows and with columns indexed by \( S \) for which the coefficients of \( N \) with \( |A| = r \) satisfy
\[
N[A] = \det N(A)
\]
where \( N(A) \) is the submatrix of \( N \) with columns \( A \), and the other coefficients are 0.
(For \( r = 0 \) the condition is \( N[\emptyset] \neq 0 \) and all other \( N[A] = 0 \).)
\( L \) is the subspace spanned by the rows of \( N \).
If \( r \neq 0 \) then \( N \) and the \( v_1, \ldots, v_r \) in \( N = v_1 \cdots v_r \) can be chosen so row \( i \) of \( N \) holds the coefficients for writing \( v_i \) as a linear combination of vectors from basis \( S \).

**Definition 2.2 (Extensor).** When the conditions in Theorem 2.1 are true about \( N \), we say \( N \) is decomposable, and the coefficients denoted by \( N[A] \) are called the Plücker coordinates of \( N \). A decomposable element of an exterior algebra is called an extensor. The integer \( r \) is called its rank and is denoted by \( \rho N \).

**Remark:** We do not define Plücker coordinates to be homogeneous, so extenders that differ by a non-zero scalar multiple have different Plücker coordinates, even though they represent the the same subspace.

**Remark:** \( N[A] \) is defined as 0 for all \( A \) with \( |A| \neq \rho N \).

The fact that each rank \( r \) extensor is the exterior product of \( r \) vectors, and \( v_1 v_2 = -v_2 v_1 \) for vectors implies that extensor multiplication satisfies the following anticommutative law:
\[
N_1 N_2 = (-1)^{\rho N_1 \rho N_2} N_2 N_1. \tag{5}
\]

**Theorem 2.3.** Suppose \( r, \) subspace \( L \), matrix \( N \), extensor \( N(S) \) and its Plücker coordinates are as described in Theorem 2.1.

1. There exists a rank \( |S| - r \) matrix \( N^\perp \) with columns indexed by \( S \) and with \( |S| - r \) rows for which coefficients with \( |A| = r \) satisfy
\[
N[A] = \det N^\perp(\overline{A}) \epsilon(\overline{AA})
\]
where \( \overline{A} = S \setminus A \) expressed in an arbitrary sequence and \( \epsilon \) is some non-zero alternating sign function of sequences over \( S \).
2. $L$ consists of all $x \in KS$ that satisfy the equations

$$N^\perp x = 0.$$ 

In other words, $L$ (the row space of $N$) and the row space of $N^\perp$ are orthogonal complements in $KS$.

**Proof.** See [28, chap. 7]. An elementary proof of theorem 2.3 related to theorem 2.1 appears in [27, VII.3 Theorem I]. The equivalence of the Grassmann-Plücker relationships to the other conditions is proved in [27, VII.6 Theorem II].

An **extensor** $N$ with **ground set** $T$ is the finite set $T$ paired with an extensor in $E(KT)$. We use the notation $N = N(T)$ and $T = S(N)$ to indicate that $N$ has ground set $T$.

We need the ground set for the same reason a ground set is necessary to define the dual of a matroid with coloops. Independent sets, or the collection of sequences $B$ for which chirotope $\chi(B) = +$ is not sufficient because a loop doesn’t appear in any of these objects. Furthermore, in our calculations and proofs we find it very helpful to combine the signs of sequences (i.e., permutations of subsets) from several different sets within one analysis without having to relabel any elements. Identities like the Tutte equations relate function values for objects with different ground sets. It is not sufficient for an extensor or chirotope to be defined up to sign for certain identities to be valid (not just up to sign). This validity may facilitate the use of the identities in computer programs.

**Theorem 2.4.** Given $N(S)$, let $N$ be a matrix satisfying Theorem 2.1.

1. The collection of those $B \subseteq S$ for which $N[B] \neq 0$ is the collection of bases of a matroid with ground set $S$.

The same matroid is presented by the independent sets of columns of $N$.

2. The function $\chi$ of sequences over $S$ into $\{+1, -1, 0\}$ for which $\chi(B)$ is the sign of $N[B]$ is the chirotope function of the oriented matroid $N$ denoted by $N = N(S) = N(N)$.

The covectors of $N$ are presented by the signatures of the supports of the $N$’s row space elements; the signed circuits (i.e., oriented matroid “vectors” with minimal support sets) are presented by the signatures of the minimal linear dependencies among the columns of $N$.

3. If $N[B] \in \{0, \pm 1\}$ for all $B$, then $N(N)$ is the unimodular (or regular) oriented matroid whose chirotope function satisfies $\chi(B) = N[B]$. Furthermore, every unimodular oriented matroid can be realized by such an $N$.

**Proof.** See [4] or [1, chap. 5]. Details pertaining to the unimodular matroids including several characterizations are given in [55, Theorem 3.1.1, p. 41].

**Definition 2.5.** If $N(S) \neq 0$ and $e \in S$ then
• $e$ is called a loop if $N[B] = 0$ for all $B$ with $e \in B$, and
• $e$ is called a coloop if every $B$ such that $N[B] \neq 0$ satisfies $e \in B$.

Remark: $e$ is therefore a loop or coloop in $N$ if and only if it is a loop or coloop respectively in the matroid presented by $N$.

**Definition 2.6.** Each $N(S) \neq 0$ defines the function $\rho_N$ on subsets $A \subseteq S$ where $\rho_N(A)$ is the rank of $A$ in the matroid presented by $N(S)$.

**Theorem 2.7.** Given $N(S) \neq 0$, $e \in S$ and $S' = S \setminus e$:

1. The Plücker coordinate function for $N$ restricted to sequences $B \subseteq S'$ is the Plücker coordinate function for an extensor denoted by $(N \setminus e)(S')$. This operation called deletion of $e$.

   $(N \setminus e) \neq 0$ if and only if $e$ is not a coloop in the matroid presented by $N$. In this case, the unoriented or oriented matroid minor $N \setminus e$ is presented by $N \setminus e$ and $\rho(N \setminus e) = \rho N$.

2. The function defined by $N[Be]$ for sequences $B \subseteq S'$ is the Plücker coordinate function for an extensor denoted by $(N/e)(S')$. This operation is called contraction of $e$.

   $(N/e) \neq 0$ if and only if $e$ is not a loop in the matroid presented by $N$. In this case, the unoriented or oriented matroid minor $N/e$ is presented by $N/e$ and $\rho(N/e) = \rho N - 1$.

Remarks: If $N = 0$ then $N/e = N \setminus e = 0$. The zero extensor $0$ does not present any matroid. Each rank 0 (empty $S$ or loops only) matroid has only one basis $\emptyset$; they are presented by the non-zero extensors $a 1(S)$ of rank 0.

**Proof.** Let $N$ be a matrix representing $N$ in Theorem 2.1.

$(N \setminus e)(S')$ is the extensor known from Theorem 2.1 when the column labelled by $e$ is deleted from $N$. Note that if $e$ is a coloop then this reduces the rank of $N$ and so $(N \setminus e)(S') = 0$.

If $e$ is a loop in $N$ then $(N/e) = 0$. Otherwise, the $N[Be]$ are fixed non-zero multiples of the Plücker coordinates from a matrix obtained from the $N$ by row operations to make all but one entry in column $e$ zero and then deleting the row and column with that non-zero entry.

See [4, §3.5] for oriented matroid minors and other structures in terms of chirotopes. □

**Theorem 2.8.** Given $N(S)$ and $e \in S$,

$$N(S) = (N/e)e + (N \setminus e)1\{e\}$$

The multiplication by $1\{e\}$ makes the ground set of the second term be $S$ instead of $S \setminus e$. It will be omitted in contexts where the ground set is clear.
Proof. Let $B \subseteq S$. We prove that each Plücker coordinate $\mathbf{N}[B]$ equals the sum of the corresponding Plücker coordinates of the extendors on the right.

If $e \in B$ we can write $B = B'e$. $\mathbf{N}[B'] = (\mathbf{N}/e)[B'] = (\mathbf{N}/e)e[B']$, and $(\mathbf{N}\setminus e)[B] = 0$. If $e \notin B$ then $(\mathbf{N}/e)e[B] = 0$ and $\mathbf{N}[B] = (\mathbf{N}\setminus e)[B]$.

It is convenient to let $\mathbf{N}/A$ denote $\mathbf{N}/a_1/\cdots/a_1$ where $A = a_1 \cdots a_k$, and similarly for $\mathbf{N}\setminus A$. It follows that $\mathbf{N}/A[X] = \mathbf{N}[XA]$ for all $X$. We note that for $\sigma \in \mathfrak{S}_k$,

$$\begin{align*}
\mathbf{N}/A_{\sigma} &= \epsilon(\sigma)\mathbf{N}/A, \text{ but} \\
\mathbf{N}\setminus A_{\sigma} &= \mathbf{N}\setminus A.
\end{align*}$$

(6)

2.3 Ground Set Orientation and Duality

Definition 2.9 (Ground set orientation). An orientation of the ground set $\epsilon$ is an alternating function into $\{+1, -1, 0\}$ of sequences of ground set elements that is non-zero on sequences of distinct elements, and which satisfies $\epsilon(\emptyset) = 1$.

One family of ground set orientations is derived from fixed linear orders on all possible ground set elements using the rule that $\epsilon(X) = (-1)^v$ where $v$ is the number of inversions in $X$ (where an inversion is $(i, j)$ with $i < j$ and $x_i > x_j$). A permutation $\sigma \in \mathfrak{S}_n$ of $\{1, \ldots, n\}$ is always considered a sequence $\sigma_1 \sigma_2 \cdots \sigma_n$ of natural numbers with ground set orientation derived from their usual ordering. Hence, $\epsilon(\sigma)$ is the usual sign of permutation $\sigma$. However, ground set orientations of matroid elements or graph edges will not be assumed to derive from a linear order.

Since permutations $\sigma \in \mathfrak{S}_n$ and sequences of ground set elements will not be confused, we will use the same symbol $\epsilon$ for permutation sign and ground set orientation.

Given a length $n$ sequence $X = x_1 \ldots x_n$ and $\sigma \in \mathfrak{S}_n$, let $X_\sigma$ denote $x_{\sigma_1} \ldots x_{\sigma_n}$. The following routine facts will be used in our proofs: Of course, $F$ is alternating means $F(X_\sigma) = \epsilon(\sigma) F(X)$ for all sequences $X$ and $\sigma \in \mathfrak{S}_{|X|}$.

Lemma 2.10. Suppose $\epsilon_1$ and $\epsilon_2$ are arbitrary alternating functions of sequences.

1. If $n = |X| = |Y|$, $\sigma \in \mathfrak{S}_n$, and $A, X, B, C, Y, D$ are sequences then

$$\begin{align*}
\epsilon_1(AXB)\epsilon_2(CYD) &= \epsilon(\sigma)\epsilon_1(AX_\sigma B)\epsilon_2(CY_\sigma D) \\
&= \epsilon(\sigma)\epsilon_1(AXB)\epsilon_2(CY_\sigma D) \\
&= \epsilon_1(AXB)\epsilon_2(CY_\sigma D).
\end{align*}$$

2. For the concatenation of sequences denoted by $XY$,

$$\begin{align*}
\epsilon_1(XY) &= (-1)^{|X||Y|}\epsilon_1(YX).
\end{align*}$$

With fixed ground set orientation $\epsilon$ in hand, we define: (Note that each ground set $S$ determines with $\epsilon$ a sign choice from among the two that both provide a presentation of the oriented matroid dual.)
Definition 2.11 (Canonical Dual). Given $N(S)$, $N^\perp[X] = N^\perp[X] = N[\bar{X}]\epsilon(\bar{X}, X)$, where $\bar{X}$ is any sequence of the distinct elements in $S \setminus X$.

The symbol $\perp_\epsilon$ will be abbreviated by $\perp$ when $\epsilon$ is irrelevant or doesn’t require emphasis.

The demonstration in [4, end of §3.6] of oriented matroid chirotope dualization has a similar formula, whose right hand side is independent of an arbitrarily chosen sequence. It follows that our extensor dualization corresponds to the oriented matroid dualization of the oriented matroid presented by $N$. Theorem 2.3 justifies this for realizable matroids. (Dualization is also the Hodge star operator [27] when $S$ is identified with the corresponding basis of the dual space. Also see §6.4.)

2.4 Identities

Our main proof uses some identities on extenders that correspond to well-known relationships among matroid operations. These identities involve extenders with ground sets for which a ground set orientation is used to define dualization. The union of disjoint sets is denoted by $\sqcup$.

Theorem 2.12.

\[ (N_1 + N_2)^\perp = N_1^\perp + N_2^\perp \]
\[ (\alpha N)^\perp = \alpha N^\perp \] (7)

\[ N^{\perp\perp}(S) = (-1)^{\rho N[|S|] - \rho N} \quad N(S) = (-1)^{\rho N\rho N^\perp} \quad N(S) \] (8)

Given $N(S)$, and sequences $X \subseteq S$ and $S' = S \setminus X$,

\[ (N \setminus X)^\perp = \epsilon(S')\epsilon(S'X) \quad (N^\perp/X) \] (9)

\[ (N/X)^\perp = \epsilon(S')\epsilon(S'X)(-1)^{|X|(|S|) - \rho N} \quad (N^\perp \setminus X) \] (10)

Given $N_i(S_i)$ with $S_1 \cap S_2 = \emptyset$, the extensor product $N_1 N_2(S_1 \sqcup S_2)$ presents the (oriented) matroid direct sum and

\[ (N_1 N_2)^\perp = \epsilon(S_1)\epsilon(S_2)\epsilon(S_1 S_2)(-1)^{\rho N_1^\perp \rho N_2} \quad N_1^\perp N_2^\perp \] (11)

Proof. Linearity (7) is immediate from the Plücker coordinate definition. It will be repeatedly used with $\alpha = \pm 1$ below.

To prove Theorem 2.12 (8), write

\[ N^{\perp\perp}[A] = (N^\perp)^\perp[A] = (N^\perp)[\overline{A}] \epsilon(\overline{A} A), \quad N[\overline{A}] \epsilon(\overline{A} A) = N[A] \epsilon(\overline{A} A) \epsilon(\overline{A} A) \]

where in the last equation we chose the sequence order $\overline{A} = A$. Therefore the sign correction is $(-1)^{|A|\overline{|A|}}$. For non-zero coordinates this is $(-1)^{\rho N[|S|] - \rho N} = (-1)^{\rho N\rho N^\perp}$. 

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To prove Theorem 2.12 (9), write
\[(N \setminus X)^{\perp}[A] = (N \setminus X)[\overline{A}] \epsilon(\overline{A}A) = N[A] \epsilon(\overline{AA}),\]
where \(\overline{A} = S' \setminus A\). But in
\[(N^\perp/X)[A] = N^\perp[AX] = N[AX] \epsilon(AXAX)\]
the elements in sequence \(AX\) are \(S' \setminus A\), the same as in the sequence symbolized by \(\overline{A}\) in the previous equation. We can therefore choose \(AX = \overline{A}\) and write
\[(N^\perp/X)[A] = N[\overline{A}] \epsilon(\overline{AA}).\]
Combining the two sign corrections gives \(\epsilon(\overline{AA}) \epsilon(\overline{AX})\). That equals \(\epsilon(S') \epsilon(S'X)\) for all reorderings \(S'\) of \(\overline{A}\).

We can get (10) from (8) and (9). Specifically, \((N/X)^{\perp} = (L^\perp/X)^{\perp}(-1)^{\rho \rho N^\perp}\) with \(L = N^\perp\). This equals
\[(L \setminus X)^{\perp\perp} \epsilon(S') \epsilon(S'X)(-1)^{\rho \rho N^\perp} = (N \setminus X)^{\perp\perp} \epsilon(S') \epsilon(S'X)(-1)^{\rho \rho N^\perp}\]
As usual, we can restrict attention to non-zero coordinates. Let \(\rho N = r\), \(|S| = s\), and \(|X| = x\) so \(\rho(N/X) = r - x\) and \(\rho(N^\perp \setminus X) = \rho N^\perp = s - r\). The sign correction is therefore
\[\epsilon(S') \epsilon(S'X)(-1)^{(s-r)(r-x)+r(s-r)} = \epsilon(S') \epsilon(S'X)(-1)^{(s-r)(2r-x)} = \epsilon(S') \epsilon(S'X)(-1)^{(s-r)x}\]

To prove Theorem 2.12 (11), take \(\overline{A_1A_2} = \overline{A_1}\overline{A_2}\) with each \(\overline{A_i} = S_i \setminus A_i\) in
\[(N_1N_2)^{\perp}[A_1A_2] = (N_1N_2)[\overline{A_1A_2}] \epsilon(\overline{A_1A_2A_1A_2}) = N_1[\overline{A_1}] N_2[\overline{A_2}] \epsilon(\overline{A_1A_2A_1A_2}) = N_1^\perp[A_1] N_2^\perp[A_2] \epsilon(\overline{A_1A_2}) \epsilon(\overline{A_1A_2A_1A_2})(-1)^{|A_1||\overline{A_2}|} = N_1^\perp[A_1] N_2^\perp[A_2] \epsilon(S_1) \epsilon(S_2) \epsilon(S_1S_2)(-1)^{\rho N_1^\perp \rho N_2^\perp}\]
where in the last equation, we applied permutations \(\sigma\) and \(\tau\), each twice, for which \((\overline{A_1A_2})\sigma = S_1\) and \((\overline{A_2A_2})\tau = S_2\). We then substituted the correct ranks for cases where the coordinate is not 0. \(\square\)

### 3 An Extensor Tutte Function

Recall that a ported extensor or matroid is one whose ground set has a distinguished subset of port elements.
Given a ported extensor $N(P, E)$ (the notation means $P$ is the set of ports and the ground set is $P \cup E$), we will define a parametrized extensor $M_E(N)$ using extensor operations. We will illustrate its construction with extensors and equivalent matrices. We will then give parametrized identities satisfied by the function $N(P, E) \to M_E(N)$ which are analogous to the ported Tutte equations. Our identities however apply to extensors rather than to commutative ring values. The identities include *sign-correction factors* that depend on the particular ground set orientation $\epsilon$ that was used to define $M_E(N)$.

The definition of $M_E(N)$ below applies to all extensors $N$ over $K(E \cup P)$. The main result therefore belongs to exterior algebra. Section 4 shows how $M_E$ defines the extensor valued function on the minor closed class of ported unimodular oriented matroids $\mathcal{N}(P, E)$ by $M_E(N) = M_E(\pm N)$ where $\mathcal{N}$ is presented by either unimodular extensor $\pm N$.

Each linear map on $V$ can be extended to a unique exterior algebra map on the exterior algebra $\mathcal{E}(V)$ [28, Theorem 7.1]. Given ground set $P \cup E$, let $P_e$ and $P_i$ be two disjoint copies of $P$. For each $p \in P$, let $p_v \in P_e$ and $p_i \in P_i$ be the corresponding elements in the respective copies. We define the following maps from $K(P \cup E)$ to $K[r_e, g_e](P_v \cup P_i \cup E)$ and extend them to the exterior algebra. If necessary, the field $K$ is extended with the parameters $g_e, r_e, \epsilon \in E$.

\[ v_r(e) = r_e e \text{ for } e \in E \text{ and } v_r(p) = p_v \text{ for } p \in P. \]
\[ t_g(e) = g_e e \text{ for } e \in E \text{ and } t_g(p) = p_i \text{ for } p \in P. \]

In terms of matrices, $t_g$ signifies multiplying column labelled $e$ by $g_e$ for each $e \in E$ and renaming column $p$ by $p_i$ for each $p \in P$. Likewise, $v_r$ signifies multiplying column $e$ by $r_e$ and renaming column $p$ by $p_v$.

The parameter subscript symbols in $t_g$ and $v_r$ will sometimes be omitted for brevity. For subset $Q \subseteq P$, $Q_v$ denotes $\{q_v : q \in Q\} \subseteq P_v$ and $Q_i$ denotes $\{q_i : q \in Q\} \subseteq P_i$.

**Definition 3.1.** Given a ported extensor $N(P, E)$, a ground set orientation $\epsilon$ and dual operator $\perp_\epsilon$, parameters $g_e$ and $r_e$ for each $e \in E$, and $\epsilon$ preserving functions $v_r$ and $t_g$ defined above, let

\[ M(N) = t_g(N) v_r(N_{\perp \epsilon}) \text{ and } M_E(N) = M(N)/E \]

Hence, $M_E(N)$ is defined as a ported extensor $M(N) = M(N)(P_i \cup P_v, E)$ contracted by the sequence of non-port elements $E$. Therefore, by Theorem 2.7 it is an extensor. Each pair of sequences $I \subseteq P$, $V \subseteq P$ with $|I| + |V| = |P|$ specifies the Plücker coordinate of $M_E(N)$ with index $I_i V_v$ and value

\[ M_E(N)[I_i V_v] = (\iota(N) v(N_{\perp \epsilon}))[I_i V_v E]. \]

**Proposition 3.2.** For $\alpha \in K$, $M_E(\alpha N) = \alpha^2 M_E(N)$.

*Proof.* $M(\alpha N) = \alpha^2 M(N)$ is immediate from the definition. Contraction $M/E$ is linear in $M$. \qed
Figure 1: Graph defining the graphic oriented matroid \( \mathcal{N} \)

We can express (13) in matrix terms. Let \( N \) be some full row rank matrix with columns indexed by \( P \cup E \) that presents \( N(P \cup E) \). Similarly, let \( N^\perp \) denote a matrix presentation of \( N^\perp \).

**Example.** We show one totally unimodular matrix representation \( N \) of the ported oriented matroid with \( P = \{p_1, p_2, p_3\} \) and \( E = \{e_1, e_2, e_3, e_4\} \) for the graph in figure 1. The rows code 3 oriented cutsets which determine a basis for the 1-coboundary (or cocycle) space. We also express \( N \) by the exterior product of the vectors given by the rows of this matrix.

\[
\begin{bmatrix}
  p_1 & p_2 & p_3 & e_1 & e_2 & e_3 & e_4 \\
  -1 & 0 & +1 & +1 & +1 & 0 & 0 \\
  0 & +1 & -1 & -1 & 0 & +1 & 0 \\
  -1 & -1 & +1 & +1 & 0 & 0 & +1 \\
\end{bmatrix}
\]

\[
N = (-p_1 + p_3 + e_1 + e_2).
\]

\[
N = (p_2 - p_3 - e_1 + e_3).
\]

\[
N = (-p_1 - p_2 + p_3 + e_1 + e_4)
\]

Next, we write one totally unimodular matrix \( N^\perp \) for the canonical dual. We have checked that the sign satisfies Definition 2.11 with \( \epsilon \) chosen so \( \epsilon(p_1 p_2 p_3 e_1 e_2 e_3 e_4) = 1 \) by verifying \( N^\perp[e_1 e_2 e_3 e_4] = N[p_1 p_2 p_3] \epsilon(p_1 p_2 p_3 e_1 e_2 e_3 e_4) \).

\[
\begin{bmatrix}
  p_1 & p_2 & p_3 & e_1 & e_2 & e_3 & e_4 \\
  0 & 0 & +1 & -1 & 0 & 0 & 0 \\
  +1 & +1 & +1 & 0 & 0 & 0 & +1 \\
  0 & +1 & +1 & 0 & -1 & 0 & 0 \\
  +1 & 0 & +1 & 0 & 0 & +1 & 0 \\
\end{bmatrix}
\]

Let \( G \) and \( R \) be the diagonal matrices of the \( g_e \) and \( r_e \). The matrix

\[
M(N) = \begin{bmatrix}
N(P) & 0 & N(E)G \\
0 & N^\perp(P) & N^\perp(E)R
\end{bmatrix}
\]

has order \((p + \epsilon) \times (2p + \epsilon)\), columns indexed by sequence \( P, P_E \) and presents \( M(\mathcal{N}) \).

**Example continued.** We abbreviate labels \( p_{i_1} \) and \( p_{v_1} \) by \( i_1 \) and \( v_1 \), etc.
\[
M(N) = \begin{bmatrix}
  i_1 & i_2 & i_3 & v_1 & v_2 & v_3 & e_1 & e_2 & e_3 & e_4 \\
-1 & 0 & +1 & 0 & 0 & 0 & g_1 & g_2 & 0 & 0 \\
0 & +1 & -1 & 0 & 0 & 0 & -g_1 & 0 & g_3 & 0 \\
-1 & -1 & +1 & 0 & 0 & 0 & g_1 & 0 & 0 & g_4 \\
0 & 0 & 0 & 0 & 0 & +1 & -r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 & +1 & 0 & 0 & 0 & r_4 \\
0 & 0 & 0 & +1 & +1 & 0 & -r_2 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 & +1 & 0 & 0 & r_3 & 0 & 0 \\
\end{bmatrix}
\]

\( M(N)(P_t \cup P_v \cup E) \) is the exterior product of the vectors in \( K(P_t \cup P_v \cup E) \) corresponding to the rows of this matrix. \( M_E(N)(P_t \cup P_v) \) appears in the expression

\[
M(N) = (M_E(N))E + \cdots
\]

where the initial term is the only one with factor \( E \).

**Example continued.** We calculate \( M_E(N) \) by doing ring operations on rows to eliminate all but one non-zero entry in each \( E \) column in \( M(N) \). The result is that

\[
g_1g_2g_3g_4^6r_2^3r_4^3M(N)
\]

is equal to the following extensor in matrix form:

\[
\begin{bmatrix}
  i_1 & i_2 & i_3 & v_1 & v_2 & v_3 & e_1 & e_2 & e_3 & e_4 \\
-r_1r_2 & 0 & r_1r_2 & 0 & g_2r_1 & g_1r_2 + g_2r_1 & 0 & 0 & 0 & 0 \\
0 & r_1r_3 & -r_1r_3 & -g_3r_1 & 0 & -g_1r_3 - g_3r_1 & 0 & 0 & 0 & 0 \\
-r_1r_4 & -r_1r_4 & r_1r_4 & -g_4r_1 & -g_4r_1 & g_1r_4 - g_4r_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_1 & -g_1r_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_3r_1 & g_3r_1 & g_4r_1 & 0 & 0 & 0 & g_4r_1r_4 \\
0 & 0 & 0 & 0 & g_2r_1 & g_2r_1 & 0 & -g_2r_1r_2 & 0 & 0 \\
0 & 0 & 0 & g_3r_1 & 0 & g_3r_1 & 0 & 0 & g_3r_1r_3 & 0 \\
\end{bmatrix}
\]

After some cancellation, we can read off the answer from the \( 3 \times 6 \) upper left submatrix, which is a matrix presentation of the extensor \( r_1^2M_E(N) \):

\[
\begin{bmatrix}
  i_1 & i_2 & i_3 & v_1 & v_2 & v_3 \\
-r_1r_2 & 0 & r_1r_2 & 0 & g_2r_1 & g_1r_2 + g_2r_1 \\
0 & r_1r_3 & -r_1r_3 & -g_3r_1 & 0 & -g_1r_3 - g_3r_1 \\
-r_1r_4 & -r_1r_4 & r_1r_4 & -g_4r_1 & -g_4r_1 & g_1r_4 - g_4r_1 \\
\end{bmatrix}
\]

**Remark:** Each Plücker coordinate of \( M_E(N) \) is a homogeneous polynomial of degree \( |E| \) in the \( g_e, r_e \). However, this example demonstrates that there sometimes doesn’t exist a matrix expression for \( M_E(N) \) all of whose entries are polynomials. The reader can verify that each order 3 minor of the above matrix is a multiple of \( r_1^2 \).
Graph Minor  Extensor Minor  Term in $M_E(N)[v_1 v_2 v_3]$

![Diagram](image)

$$N/\{e_1, e_4\}|P = -g_1 g_4 r_2 r_3$$

$$N/\{e_2, e_3\}|P = +g_2 g_3 r_1 r_4$$

Figure 2: The two graph and extensor minors with corresponding terms in $M_E(N)[v_1 v_2 v_3]$.

**Example continued:** Here are examples of Plücker coordinates, which can be calculated from the above matrix as order 3 minors divided by $r_i^2$. See §6.3.

$$M_E(N)[v_1 v_2 v_3] = g_1 g_2 g_3 r_4 + g_1 g_2 g_4 r_3 + g_1 g_3 g_4 r_2 + g_2 g_3 g_4 r_1$$

$$M_E(N)[i_1 i_2 i_3] = (g_1 r_3 + g_3 r_1)(g_2 r_4 + g_4 r_2)$$

$$M_E(N)[v_1 v_2 v_3] = -g_1 g_4 r_2 r_3 + g_2 g_3 r_1 r_4$$

Observe $M_E(N)[v_1 v_2 v_3]$ is the basis enumerator for $N(N \setminus P$. The graph and extensor minors corresponding to the terms of $M_E(N)[v_1 v_2 v_3]$ are shown in figure 2.

### 3.1 Main Result

**Theorem 3.3.** The parametrized extensor valued function $M_E(N)(P \cup P')$ of ported extensor $N = N(P, E)$ has the following properties:

1. Given $N_1(P_1, E_1)$ and $N_2(P_2, E_2)$ with $E = E_1 \cup E_2$ and $P = P_1 \cup P_2$,

$$M_E(N_1 N_2)(P, E) = \epsilon(P_1 P_2 E)\epsilon(P_1 E_1)\epsilon(P_2 E_2) M_{E_1}(N_1) M_{E_2}(N_2).$$  \hspace{1cm} (16)

2. If $e \in E$ and $E' = E \setminus e$ then

$$M_E(N) = \epsilon(PE)\epsilon(P'E') (g, M_{E'}(N/e) + r_e M_{E'}(N \setminus e)).$$ \hspace{1cm} (17)

3. Let $E = \emptyset$. The Plücker coordinates of $M_0(N)(P \cup P_v)$ satisfy

$$M_0(N)[I_v V_v] = M[I_v V_v] = \epsilon(\overline{V} V) N[I]\overline{N[N[V].$$

for all $I \subseteq P$ and $V \subseteq P$.

4. $M_E(0) = 0$. 

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Properties 1. and 2. in terms of Plücker coordinates are:

\[
M_E(N_1 \cap N_2)[I_{1V_1}I_{2V_2}] = \
\epsilon(P_1P_2E)\epsilon(P_1E_1)\epsilon(P_2E_2) M_{E_1}(N_1)[I_{1V_1}] M_{E_2}(N_2)[I_{2V_2}].
\]

(18)

\[
M_E(N)[I_{1V_1}] = \
\epsilon(PE)\epsilon(PE') ((g_e M_{E'}(N/e)[I_{1V_1}] + r_e M_{E'}(N \setminus e)[I_{1V_1}]).
\]

(19)

Remarks:

1. Property 2. implies that every linear combination \( g_e M_{E'}(N/e) + r_e M_{E'}(N \setminus e) \) is decomposable, i.e., an extensor.

2. Proposition 3.2 with \( \alpha = \pm 1 \) implies \( M_E(N_1N_2) = M_E(N_2N_1) \). We can also verify this from the right hand side of property 1 using Lemma (2.10.3) and noting the rank of extensor \( M_{E_i}(N_i) \) is \( |P_i| \).

3. If \( N \neq 0 \), one but not both of \( N/e \) and \( N \setminus e \) will be the 0 extensor if and only if \( \epsilon \) is a loop or a coloop in the matroid of \( N \). If \( N' = 0 \) then \( N'^{\perp} = 0 \) and \( M_E(N') = 0 \). We therefore write property 2 without restricting \( \epsilon \) to a non-separator.

4. Property 1 except for signs is immediate from direct sum of subspaces and their corresponding extendors. Property 2 except for the signs follows immediately from the fact that minor \( [I_{1V_1}E] \) of matrix (15) equals a linear combination with coefficients \( g_e \) and \( r_e \) because the column \( e \) belongs to this minor no matter which \( e \in E \) is specified for the identity.

**Proof.** From the definition \( M(N_1N_2) = \iota(N_1N_2) \iota((N_1N_2)^+) \) which equals

\[
\epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)(-1)^{\rho N_1^+ \rho N_2} \iota(N_1N_2) \iota((N_1N_2)^+)
\]

by Theorem 2.12(8). \( \iota(N_1N_2) \iota((N_1N_2)^+) = \iota(N_1)\iota(N_2)\iota((N_1^+)\iota(N_2^+) \) which equals (by (5))

\[
(-1)^{\rho N_1^+ \rho N_2} \iota(N_1) \iota(N_1^+) \iota(N_2) \iota(N_2^+).
\]

Therefore \( M(N_1N_2) = \epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2) \iota(N_1) \iota(N_1^+) \iota(N_2) \iota(N_2^+) \).

(20)

Therefore, \( M(N_1N_2)/E_1E_2 = \epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2) (M(N_1)/E_1 E_1) (M(N_2)/E_2 E_2) / E_1E_2 \)

because the only Plücker coordinates of the form \( M(N_i)[X_i] \) for \( i = 1 \) or 2 that contribute to (20) when it is contracted by \( E_1E_2 \) satisfy \( E_i \subseteq X_i \). The anticommutativity law (5) then implies that \( M(N_1N_2)/E_1E_2 = \epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)(-1)^{|E_1||E_2|} (M_{E_1}(N_1) M_{E_2}(N_2) E_1 E_2) / E_1E_2 \)

\[
eq \epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)(-1)^{|E_1||E_2|} M_{E_1}(N_1) M_{E_2}(N_2).
\]
Since the orders of the $S_i$ are arbitrary, let $S_i = P_i E_i$ for $i = 1$ and 2. According to equation (6), $M_E(N_1 N_2) = M(N_1 N_2)/E = \epsilon(\sigma) M(N_1 N_2)/E_1 E_2$ where $(E_1 E_2) = E$. Therefore, $M_E(N_1 N_2) = \pm M_{E_1}(N_1) M_{E_2}(N_2)$ with the sign equal to

$$
\epsilon(\sigma)\epsilon(P_1 E_1)\epsilon(P_2 E_2)\epsilon(P_1 E_1 P_2 E_2)(-1)^{|E_1||P_2|}
$$

$$
= \epsilon(\sigma)\epsilon(P_1 E_1)\epsilon(P_2 E_2)\epsilon(P_1 P_2 E_1 E_2)
$$

$$
= \epsilon^2(\sigma)\epsilon(P_1 E_1)\epsilon(P_2 E_2)\epsilon(P_1 P_2 (E_1 E_2)\sigma)
$$

$$
= \epsilon(P_1 E_1)\epsilon(P_2 E_2)\epsilon(P_1 P_2 E),
$$

which proves property 1 of the theorem.

Now for property 2. Let us apply Theorem 2.8 to $N$ and $N^\perp$, and apply $\iota$ and $\nu$ respectively.

$$
N = (N/e)e + (N \setminus e)1(e).
$$

$$
N^\perp = (N^\perp/e)e + (N^\perp \setminus e)1(e).
$$

$$
\iota(N) = \iota((N/e)e) + \iota((N \setminus e)1(e))
$$

$$
= \iota(N/e)g_e e + \iota(N \setminus e)1(e).
$$

$$
\nu(N^\perp) = \nu((N^\perp/e)e) + \nu((N^\perp \setminus e)1(e))
$$

$$
= \nu(N^\perp/e)r_e e + \nu(N^\perp \setminus e)1(e).
$$

The exterior product of $\iota(N)$ and $\nu(N^\perp)$ is therefore

$$
g_e \iota(N/e) e \nu(N^\perp \setminus e) + r_e \iota(N \setminus e) \nu(N^\perp/e) e
$$

$$
+ g_e r_e \iota(N/e) e \nu(N^\perp/e) e
$$

$$
+ \iota(N \setminus e) \nu(N^\perp \setminus e).
$$

$M_E(N)$ is the result of contracting the above by $E$. The third term above is 0 because $e$ is a repeated factor. The last term will vanish when contracted by $E$ because none of its non-zero Plücker coordinates have an index that contains $e$. So we will omit them in the following. By Theorem 2.12(10)

$$
\nu(N^\perp \setminus e) = \nu((N/e)^\perp) \epsilon(S')\epsilon(S'\epsilon)(-1)^{|S'|-\rho N}
$$

and

$$
\nu(N^\perp/e) = \nu((N \setminus e)^\perp) \epsilon(S')\epsilon(S'\epsilon).
$$

Notice that $\rho(N/e^\perp) = |S'| - (\rho N - 1) = |S| - \rho N$. So $e \nu(N/e^\perp) = \nu(N/e^\perp) e (-1)^{|S| - \rho N}$. Therefore when the above substitutions are made we get

$$
M_E(N)\epsilon(S')\epsilon(S'\epsilon) = (g_e \iota(N/e) \nu((N/e)^\perp) + r_e \iota(N \setminus e) \nu((N \setminus e)^\perp)) e/E.
$$

Lemma 2.10 used with $\sigma$ such that $E_\sigma = E'e$ shows that the right hand side is

$$
\epsilon(\sigma)((\cdots)e)/E'e = \epsilon(E)\epsilon(E'e)((\cdots)e)/E'e.
$$
So the right hand side is
\[
\epsilon(E)\epsilon(E'e)(g_eM_{E'}(N/e) + r_eM_{E'}(N \setminus e)).
\]
Since the order of \( S' \) is arbitrary, we can choose \( S' = PE' \). The sign correction is then
\[
\epsilon(S')\epsilon(S'e)\epsilon(E)\epsilon(E'e) = \\
\epsilon(PE')\epsilon(PE'e)\epsilon(E)\epsilon(E'e).
\]
Applying the \( \tau \) for which \( (E'e)_{\tau} = E \) to the two appearances of subsequence \( E'e \) doesn’t change this expression’s value. Hence the sign correction is
\[
\epsilon(PE')\epsilon(PE)\epsilon(E)\epsilon(E) = \epsilon(PE')\epsilon(PE)
\]
and property 2 of the theorem is verified.

The definition of \( M_{E'} \) immediately gives property 4, and, together with the definition of extensor dual, gives property 3.

**Corollary 3.4.** The set of \( M_{E'}(N) \) obtained as the \( g_e, r_e \) range over \( \mathbb{R} \) for each \( e \in E \) represents the points in a projective subspace of a Grassmannian (which consists of all the linear subspaces over \( \mathbb{R}(P_i \cup P_j) \) with dimension \( |P| \)).

**Proof.** Induction: Use Theorem 3.3 property 2 for when \( |E| > 0 \) and property 3 for the the basis.

**Proposition 3.5.** Given \( N = N(P,E) \), and sequences \( I \subseteq P, V \subseteq P, \) and \( \bar{V} = P \setminus V \),
\[
\epsilon(\bar{V}V)e(PE)M_{E}(N)[I_iV_v] = \epsilon(P) \sum_{A \subseteq E} N[A|N[\bar{V}A]g_Ar_{\tau}.
\]

**Remark:** The only non-zero terms in this sum are those for which both \( A \cup I \) and \( A \cup \bar{V} \) are bases in the matroid of \( N \).

**Proof.** Recalling that \( E \) symbolizes a sequence \( e_1 \cdots e_n \), let \( E_i = e_iE_{i+1} \) so \( E_1 = E \) and \( E_{n+1} = \emptyset \). When property (2) of Theorem 3.3 is applied successively for \( \epsilon := e_i, E := E_i \) and \( E' := E_{i+1} \) for \( i = 1, 2, \ldots, n \) and the products are expanded, the result is a sum of \( 2^n \) terms, one for each subset \( A \subseteq E \). For each fixed \( i, 1 \leq i \leq n \), the instances of the symbols \( E \) and \( E' = E \setminus e_i \) within all applications of property (2) each denote the same sequences. Therefore, sign cancellation occurs and we can write
\[
M_{E}(N)[I_iV_v] = \epsilon(PE) \sum_{A \subseteq E} \epsilon(P)g_Ar_{\tau}M_\emptyset(N/A \setminus \emptyset)[I_iV_v].
\]

Property (3) combined with the definitions of extensor deletion, contraction and dualization demonstrate that within each term
\[
M_\emptyset(N/A \setminus \emptyset)[I_iV_v] = \epsilon(\bar{V}V)N[I_A|N[\bar{V}A],
\]
and the conclusion follows.

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Corollary 3.6. If $N \neq 0$ and the parameters are generic or are all positive, then $M_E(N) \neq 0$.

Proof. $N$ has some basis, i.e., $B \subseteq P \cup E$ for which $N[B] \neq 0$. (N.B. $B = \emptyset$ is possible.) Take $I = B \cap P$ and $V = P \setminus I$. Proposition 3.5 indicates $M_E(N)[I, V] \neq 0$ with $A = B \setminus I$. \hfill \Box

Corollary 3.7. Plücker coordinate $M_E(N)[I, V]$ is a homogeneous polynomial in the $g_e$, $r_e$ whose terms are square-free and have degree $\rho N - |I|$ in the $g_e$ and degree $|E| - \rho N + |I| = |E| + |P| - \rho N - |V| = \rho N^+ - |V|$ in the $r_e$.

Proof. Immediate from Proposition 3.5, matroid duality and the fact $N[X] \neq 0$ only if $|X| = \rho N$. \hfill \Box

Corollary 3.8.

$$\epsilon(PE)M_E(N) = \epsilon(P) \sum_{A \subseteq E : \rho N^A = |A|, \rho N - \rho(N/A|P) - \rho N^A = 0} M_0(N/A|P)g_A\pi_T. \quad (22)$$

Proof. The definition of extensor deletion indicates $N/A|P$ is an alternative notation for $N/A \setminus \hat{A}$ when $A \subseteq E$, $\hat{A}$ means $E \setminus A$ and the ground set of $N$ is $P \cup E$. The conditions stated in Corollary 3.7 allow us to restrict the sum as indicated. Hence formula (21) for each Plücker coordinate is equivalent to the given expression for the extensor. \hfill \Box

The following definition and consequence of Proposition 3.5 clarify some of the sign behavior resulting from the definitions.

Definition 3.9. A function $F = F^\epsilon(X)$ whose value might depend on the ground set orientation $\epsilon$ and on the sequence $X$ is said to be

1. alternating in $X$ if $F^\epsilon(X_\sigma) = \epsilon(\sigma)F^\epsilon(X)$, for all $\sigma \in S_{|X|}$; and
2. alternating in $\epsilon$ if $F^{-\epsilon}(X) = -F^\epsilon(X)$.

Corollary 3.10. Let $Q \subseteq P_i \cup P_v$ with $|Q| = |P|$.

1. $M_E^\epsilon(\pm N)[Q]$ is constant under sign change of $\pm N$, and is alternating in $E$, $\epsilon$ and $Q$.
2. $\epsilon(PE)M_E^\epsilon(\pm N)[Q]$ is constant under sign change of $\pm N$ and under changes or reorderings of $\epsilon$ or $E$; it is alternating in $P$ and in $Q$.
3. $\epsilon(PE)M_E^\epsilon(\pm N)[P_i]$ enumerates the bases of $N(N/P)$, assuming $P$ is independent in the matroid $N(N)$, by

$$\epsilon(PE)M_E^\epsilon(\pm N)[P_i] = \sum_{B \subseteq E} g_B\pi_TN^2\|BP \|.$$

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4. and $\epsilon(PE)\mathcal{M}_E^P(\pm N)[P_v]$ enumerates the bases of $\mathcal{N}(N \setminus P)$, assuming $P$ is coindependent in $\mathcal{N}(N)$, by

$$
\epsilon(PE)\mathcal{M}_E^P(\pm N)[P_v] = \sum_{B \subseteq E} g_B r_{\mathcal{H}^2}[B].
$$

**Remark:** Properties 3. and 4. express the Matrix Tree Theorem.

## 4 Corank-Nullity Polynomials

The well-known corank-nullity (or rank) polynomial is easily generalized to ported matroids. We did [11] this with a definition that differs from Las Vergnas’ big Tutte polynomial [32] only in notation and in our applications. We will now generalize it by adding parameters and modify it for oriented matroids by reinterpreting the symbols for minors. In the definition below the symbol $\mathcal{N}/A|P$ represents the oriented minor of the oriented matroid $\mathcal{N}$ obtained by contracting $A$ and restricting to $P$. Minors that are different orientations of the same unoriented matroid are deemed different objects.

**Definition 4.1 (Parametrized and Ported Corank-Nullity Polynomial).**

$$
R(\mathcal{N}(P,E)) = \sum_{A \subseteq E} [\mathcal{N}/A|P] g_A r_{\mathcal{H}} u^{\rho_N - \rho[\mathcal{N}/A|P] - \rho A} v^{\rho A - \rho A}.
$$

In this formula, the bracketed oriented matroid $[\mathcal{N}/A|P] = [N_{i_1} \oplus \ldots \oplus N_{i_e}]$ denotes the (commutative) product of the variables $[N_{i_1}], \ldots, [N_{i_e}]$, where each variable signifies a connected component of $\mathcal{N}/A|P$. If $P = \emptyset$ then $[\mathcal{N}/A|P] = [\emptyset] = 1$; so $R$ reduces to the corank-nullity polynomial, parametrized.

The formula therefore defines a polynomial in parameters $g_e, r_e$, whose (other) variables are $u, v$ together with a distinct variable for every connected component of every minor of $\mathcal{N}$ obtained by contracting some subset $A \subseteq E$ and deleting $\overline{A} = E \setminus A$. The latter variables only occur in monomials that signify direct sums of one or more minors.

It is readily verified that $R(\mathcal{N}(P,E))$ satisfies the ported Tutte equations below. The details published in [11] can be immediately adapted to the changes we described above.

We state these results without proof:

**Proposition 4.2.** 1. If $e \in E$ is neither a port nor a loop nor a coloop in $\mathcal{N}(P,E)$,

$$
R(\mathcal{N}(P,E)) = g_e R(\mathcal{N}/e) + r_e R(\mathcal{N} \setminus e).
$$

2. $R(\mathcal{N}_1 \oplus \mathcal{N}_2) = R(\mathcal{N}_1) R(\mathcal{N}_2) = R(\mathcal{N}_2) R(\mathcal{N}_1)$.

3. $R(\mathcal{N}_1(e)) = g_e + r_e u$ and $R(\mathcal{N}_0(e)) = r_e + g_e v$, for the coloop and loop matroids $\mathcal{N}_1(e)$ and $\mathcal{N}_0(e)$ on $E = \{e\}$, $P = \emptyset$.

4. $R(\mathcal{N}(P,\emptyset)) = [\mathcal{N}]$ (i.e. when $E = \emptyset$.)
Let us take $N$ to be the oriented matroid presented by extensor $N$. The reader can now verify that the expansion in corollary 3.8 for $M_E(N)$ is obtained from $R(N(P,E))$ by substituting $u = 0$, $v = 0$ and the extensor $\epsilon(P)\epsilon(PE)M_0(N/A|P)$ for the monomial $[N/A|P]$ in the term with factor $g_A^r \tau_A$, for each $A$. Note that $|A| - \rho A = 0$ and $\rho N - \rho[N/A|P] - \rho A = \rho N - \rho (P \cup A) = 0$ imply that $A$ is independent and $P \cup A$ is spanning in $N$. Therefore $N/A|P \neq 0$ for those terms where the exponents of $u$ and $v$ are both zero.

With arbitrary $N$, the substitution of $\epsilon(PE)\epsilon(P)M_0(N/A_i|P)$ for monomial $[N/A_j|P] = [N/A_i|P]$ in $R(N(P,E))$ is not well-defined. The reason is that the same oriented matroid $N/A_j|P$ might be represented by different extensors all with the form $N/A_i|P$, for various $A_i \neq A_j$. They may differ by representing different subspaces (in the same oriented matroid stratification [4, §2.4] layer) of $KP$. Therefore, different values $M_E(N/A_i|P)$ must be substituted in $[N/A_i|P]g_A^r \tau_A$ with different $A_i$ even though these $[N/A_i|P]$ all denote the same oriented matroid.

The one general situation where $R(N)$ with $u = v = 0$ determines $M_E(N)$ is when $N(P,E)$ is a unimodular extensor, i.e., one that represents the unimodular oriented matroid $N$. [55, Theorem 3.1.1, p. 41] provides this among other equivalent characterizations of unimodular (also called regular) matroids. One of these characterizations is that bracket values from $\{+1, -1, 0\}$ may be assigned so the Grassmann-Plücker relationships hold over $Q$.

**Definition 4.3.** The extensor-valued function $N(P,E) \to M_E(N)$ is defined on the minor-closed class of ported unimodular oriented matroids by

$$M_E(N) = M_E(N)$$

where $\pm N(P,E)$ are the two unimodular presentations of the port oriented matroid $N(P,E)$.

The simplest case to demonstrate that the unimodular matroids must be oriented in order for monomial substitution in $R(N(P,E))$ to produce $M_E(N)$ is the two orientations $N_1, N_2$ of the 2-circuit matroid on two ports. Here, $E = \emptyset$ and $M_0(N_1) \neq M_0(N_2)$.

We conclude:

**Theorem 4.4.** For unimodular oriented matroid $N = N(P,E)$, $M_E(N)$ is the result of evaluating $R(N)$ (in the exterior algebra) after the substitutions $u = 0$, $v = 0$ and extensor $\epsilon(PE)\epsilon(PE)M_0(N(P))$ for each monomial $[N_i(P)]$ (which symbolizes an oriented unimodular matroid with ground set $P$).

**Proof.** Immediate from the above remarks and Corollary 3.8. \[\square\]

**Theorem 4.5.** The function $M_E(N)$ defined above on ported unimodular oriented matroids satisfies the properties: (Symbols like $E$ and $P$ denote sequences and $M_E(N)$ depends on the $g_e, r_e$ and $\epsilon$.)

1. If $e \in E$ is neither a separator nor a port, and $E' = E \setminus E$, then

$$M_E(N) = \epsilon(PE)\epsilon(PE') \left(g_eM_{E'}(N/e) + r_eM_{E'}(N \setminus e)\right).$$

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2. If \( \mathcal{N}_1(P_1, E_1) \) and \( \mathcal{N}_2(P_2, E_2) \) have disjoint ground sets and \( E = E_1 \cup E_2 \), then
\[
M_E(\mathcal{N}_1 \oplus \mathcal{N}_2) = \epsilon(P_1P_2E)\epsilon(P_1E_1)\epsilon(P_2E_2) M_{E_1}(\mathcal{N}_1) M_{E_2}(\mathcal{N}_2).
\]

3. If \( P = \emptyset \) and \( \mathcal{B}(\mathcal{N}) \) denotes the collection of bases of \( \mathcal{N} \), then
\[
M_E(\mathcal{N}) = \epsilon(E) \sum_{B \in \mathcal{B}(\mathcal{N})} g_B r_B
\]

Proof. Immediate from Theorem 3.3 and the above remarks.

5 Basis, Set and Flat Expansions

Theorem 4.5 shows that when \( \mathcal{N}(P, E) \) is a unimodular matroid, \( M_E(\mathcal{N}) \) is a substitution of extensors and \( u = v = 0 \) into \( R_P(\mathcal{N}) \) of Definition 4.1. Proposition 4.2 demonstrates \( R_P(\mathcal{N}) \) is a ported Tutte function. This motivates general study of ported and parametrized Tutte functions of matroids. We will extend some known general expressions for Tutte functions and invariants, and identify an unsolved problem due to combining parameters with ports.

Suppose a function \( F \) is defined on a minor closed class of unoriented or oriented matroids, and \( F \) satisfies the ported Tutte equations with given parameters \( g_e, r_e \) for the non-port elements. In other words, \( F \) is a ported Tutte function. It is apparent that for each \( \mathcal{N} \) in the class, \( F(\mathcal{N}) \) is determined uniquely by the parameters together with the values \( F(\mathcal{N}_0(e)), F(\mathcal{N}_1(e)) \) (called point values) and \( F(\mathcal{N}_k) \) on particular loops, coloops, and the other indecomposable (or empty) \( \mathcal{N}_k \) in the class. The \( \mathcal{N}_k \) are the connected matroids in the class, including the empty matroid, whose ground sets consist only of ports.

Zaslavsky [57] observed that when arbitrary parameters and point values are prescribed, the Tutte function might not exist on some (non-ported) matroids and their minors. This contrasts with the fact that Tutte invariants are always uniquely determined, via the non-parametrized Tutte polynomial, from the two point values for the isomorphism classes of loop and coloop matroids. Zaslavsky’s results (1) classify Tutte functions by classifying solutions to constraints on prescribed parameters and point values necessary for a Tutte function to exist, and (2) demonstrate that each Tutte function value is given by the basis expansion expressions (i.e., their common value), for each class member, derived from all the ground set element orderings. See also Bollobas and Riordan’s paper [5] which address different problems with a similar theme, and the unified explanation of the subject by Ellis-Monaghan and Traldi [21].

Zaslavsky defines the normal class as the one for which there exist \( u \) and \( v \) for which, for all \( e \), the point value on coloop \( e \) is \( r_e u + g_e \) and the point value on loop \( e \) is \( g_e v + r_e \). Only normal Tutte functions are obtained by substitutions into the parametrized corank-nullity polynomial. Our Theorem 4.4 expresses how \( M_E(\mathcal{N}) \) fits into the natural ported generalization of the normal class.
While only the normal Tutte functions have corank-nullity polynomial expressions, they all have basis expansion expressions. In the rest of this section, we will discuss these and other expansion expressions for ported unoriented and oriented matroids.

The basis expansion originated by Tutte [43] for graphs and Crapo [19] for matroids depends on a particular but arbitrary ground set element order $O$. Each basis determines a term from the internal and external activities of elements with respect to that basis according to the ordering $O$. Our way to generalize is to restrict $O$ to orders in which every port element is ordered before each $e \in E$. (We use the convention that the deleted/contracted element is the last, i.e., greatest element under order $O$ eligible for reduction.)

Gordan and McMahon define [25] a “computation tree” to formalize the application of a subset of Tutte equations to a matroid and some of its minors. Each (Tutte) computation tree determines a polynomial in the parameters and point values. Therefore, when $\mathcal{N}$ is in the domain of a Tutte function, each of these computation trees determine the same value. Computation trees are a way to give a basis expansion expression in terms of a more general definition of internal and external activities of elements with respect to a basis. The expansion is more general because it is based on any Tutte equation computation rather than on an element order $O$. We will extend to ported computation trees the classification [25] of elements as internally or externally, active or passive with respect to each path down the tree. In each case, the result is an interval partition of the boolean subset lattice of $E$.

**Definition 5.1.** Given $\mathcal{N}(P, E)$, a $P$-subbasis $F \in \mathcal{B}_P(\mathcal{N})$ is an independent set with $F \subseteq E$ (so $F \cap P = \emptyset$) for which $F \cup P$ is a spanning set for $\mathcal{N}(P, E)$ (in other words, $F$ spans $\mathcal{N}/P$, see [32].)

**Proposition 5.2.** For every $P$-subbasis $F$ there exists an independent set $Q \subseteq P$ that extends $F$ to a basis $F \cup Q \in \mathcal{B}(\mathcal{N})$. Conversely, if $B \in \mathcal{B}(\mathcal{N})$ then $F = B \cap E = B \setminus P$ is a $P$-subbasis.

**Proof.** Immediate. \hfill $\square$

**Definition 5.3 (Activities with respect to a $P$-subbasis and an element ordering $O$).** Let ordering $O$ have every $p \in P$ before every $e \in E$. Let $F$ be a $P$-subbasis. Let $B$ be any basis for $\mathcal{N}$ with $F \subseteq B$.

- Element $e \in F$ is internally active if $e$ is the least element within its principal cocircuit with respect to $B$. Thus, this principal cocircuit contains no ports. The reader can verify this definition is independent of the $B$ chosen to extend $F$. Elements $e \in F$ that are not internally active are called internally inactive.

- Dually, element $e \in E$ with $e \not\in F$ is externally active if $e$ is the least element within its principal circuit with respect to $B$. Thus, each externally active element is spanned by $F$. Elements $e \in E \setminus F$ that are not externally active are called externally inactive.
Definition 5.4 (Computation Tree, following [25]). A ported (Tutte) computation tree for $\mathcal{N}(P, E)$ is a binary tree whose root is labelled by $\mathcal{N}$ and which satisfies:

1. If $\mathcal{N}$ has non-separating elements not in $P$, then the root has two subtrees and there exists one such element $e$ for which one subtree is a computation tree for $\mathcal{N}/e$ and the other subtree is a computation tree for $\mathcal{N} \setminus e$.

   The branch to $\mathcal{N}/e$ is labelled with “$e$ contracted” and the other branch is labelled “$e$ deleted”.

2. Otherwise (i.e., every element in $S(\mathcal{N}) \setminus P$ is separating) the root is a leaf.

An immediate consequence is

Proposition 5.5. Each leaf of a $P$-ported computation tree for $\mathcal{N}(P, E)$ is labelled by the direct sum of some minor of $\mathcal{N}$ on $P$ (oriented if $\mathcal{N}$ is oriented) summed with loop and/or coloop matroids with ground sets $\{e\}$ for various distinct $e \in E$ (possibly none).

Definition 5.6 (Activities with respect to a leaf). For a ported computation tree for $\mathcal{N}(P, E)$, a given leaf, and the path from the root to this leaf:

- Each $e \in E$ labelled “contracted” along this path is called **internally passive**.
- Each coloop $e \in E$ in the leaf’s matroid is called **internally active**.
- Each $e \in E$ labelled “deleted” along this path is called **externally passive**.
- Each loop $e \in E$ in the leaf’s matroid is called **externally active**.

Proposition 5.7. Given a leaf of a ported computation tree for $\mathcal{N}(P, E)$: The set of internally active or internally passive elements constitute a $P$-subbasis of $\mathcal{N}$ which we say **belongs to the leaf**. Furthermore, every $P$-subbasis $F$ of $\mathcal{N}$ belongs to a unique leaf.

Proof. For the purpose of this proof, let us extend Definition 5.6 so that, given a computation tree with a given node $i$ labelled by matroid $\mathcal{N}_i$, $e \in E$ is called internally passive when $e$ is labelled “contracted” along the path from root $\mathcal{N}$ to node $i$. Let $IP_i$ denote the set of such internally passive elements.

It is easy to prove by induction on the length of the root to node $i$ path that (1) $IP_i \cup S(\mathcal{N}_i)$ spans $\mathcal{N}$ and (2) $IP_i$ is an independent set in $\mathcal{N}$. The proof of (1) uses the fact that elements labelled deleted are non-separators. The proof of (2) uses the fact that for each non-separator $f \in \mathcal{N}/IP_i$, $f \cup IP_i$ is independent in $\mathcal{N}$.

These properties applied to a leaf demonstrate the first conclusion, since each $e \in E$ in the leaf’s matroid must be a separator by Definition 5.4.

Given a $P$-subbasis $F$, we can find the unique leaf as follows: Beginning at the root, descend the tree according to the rule: At each branch node, descend along the edge labelled “$e$-contracted” if $e \in F$ and along the edge labelled “$e$-deleted” otherwise (when $e \not\in F$). (This algorithm also operates on arbitrary $F' \subseteq E$.)
The above definitions and properties enable us to conclude:

**Proposition 5.8.** Given element ordering $O$ in which every $p \in P$ is ordered before each $e \notin P$, suppose we construct the unique $P$-ported computation tree $T$ in which the greatest non-separator $e \in E$ is deleted and contracted in the matroid of each tree node.

The activity of each $e \in E$ relative to ordering $O$ and $P$-subbasis $F \subseteq E$ is the same as the activity of $e$ defined with respect to the leaf belonging to $F$ in $T$.

**Definition 5.9.** Given a computation tree for (oriented) matroid $\mathcal{N}(P, E)$, each $P$-subbasis $F \subseteq E$ is associated with the following subsets of non-port elements defined according to Definition 5.6 from the unique leaf determined by the algorithm given above.

- $IA(F) \subseteq F$ denotes the set of internally active elements,
- $IP(F) \subseteq F$ denotes the set of internally passive elements,
- $EA(F) \subseteq E \setminus F$ denotes the set of externally active elements, and
- $EP(F) \subseteq E \setminus F$ denotes the set of externally passive elements.

**Proposition 5.10.** Given a computation tree for $\mathcal{N}(P, E)$, the boolean lattice of subsets of $E$ is partitioned by the collection of intervals $[IP(F), F \cup EA(F)]$ (note $F \cup EA(F) = IP(F) \cup A(F)$) determined from the collection of $P$-subbases $F$, which correspond to the leaves.

**Proof.** Every subset $F' \subseteq E = S(\mathcal{N}) \setminus P$ belongs to the unique interval corresponding to the unique leaf found by the tree descending algorithm given at the end of the previous proof.

Dualizing, we obtain:

**Proposition 5.11.** Given a computation tree for $\mathcal{N}(P, E)$, the boolean lattice of subsets of $E$ is also partitioned by the collection of intervals $[EP(F), E \setminus F \cup IA(F)]$ (note $E \setminus F \cup IA(F) = EP(F) \cup A(F)$).

**Proof.** The dual of the tree descending algorithm is to descend along the edge labelled “$e$-deleted” if $e \in F$.

The following generalizes the basis expansion expression given in [57] to ported (oriented) matroids, as well as Theorem 8.1 of [32].

**Definition 5.12.** Given parameters $g_e, r_e$, point values $x_e, y_e$, and (oriented) $\mathcal{N}(P, E)$ the Tutte polynomial expression determined by the sets in Definition 5.9 from a computation tree is equal to

$$
\sum_{F \in B_P} [\mathcal{N}/F|P] x_{IA(F)} g_{IP(F)} y_{EA(F)} r_{EP(F)}.
$$

(24)

Each Tutte polynomial expression is constructed by applying some of the Tutte equations. Therefore, if $\mathcal{N}(P, E)$ is in the domain of Tutte function $f$, then $f(\mathcal{N})$ is given by any Tutte polynomial expression with $f(\mathcal{N}/F|P)$ substituted for each oriented or unoriented matroid monomial $[\mathcal{N}/F|P]$. (This generalizes the expression used in [57] to define the Tutte polynomial under the condition that all expansions yield the same expression.)
5.1 Boolean Interval Expansion

The following proposition expresses the ported corank-nullity polynomial in terms of a $P$-subbasis expansion. It is obtained by substituting binomials $x_e = g_e + r_e u$, $y_e = r_e + g_e v$ and the matroid variables for themselves into Definition 5.12. The different expansions from different element orderings and Tutte computation trees all express the same polynomial because Proposition 4.2 demonstrates that $R_P$ is a ported Tutte function and the values of $R_P$ on coloop, loop and indecomposable matroids are readily verified to be given by these substitutions.

Proposition 5.13. The polynomial $R_P(\mathcal{N})$ is given by the following activities and boolean interval expansion formula:

$$R_P(\mathcal{N}) = \sum_{F \in \mathcal{B}_P} [\mathcal{N}/F|P] \left( \sum_{IP(F) \subseteq K \subseteq F \text{ and } EP(F) \subseteq L \subseteq E \setminus F} g_{K \cup (E \setminus F \setminus L)} v^{[E \setminus F \setminus L]} r_{L \cup (F \setminus K)} u^{[F \setminus K]} \right) \quad (25)$$

Proof. Let $A = K \cup (E \setminus F \setminus L)$ within the above expansion. We can verify $\overline{A} = E \setminus A = L \cup (F \setminus K)$. For each $A \subseteq E$ a unique $P$-subbasis $F$, and two tree leaves are determined, one by the tree descending algorithm and the other by the dual algorithm. Thus $A$ and $\overline{A}$ respectively belong to intervals within the boolean lattice partitions of Propositions 5.10 and 5.11. In particular, $A \in [IP(F), F \cup EA(F)]$ and $\overline{A} \in [EP(F), E \setminus F \cup IA(F)]$. Therefore the terms in the above sum are equal one by one to the terms in the corank-nullity polynomial’s subset expansion (Definition 4.1).

For the purposes of this paper it was sufficient to recognize that our extensor valued ported parametrized Tutte function of unimodular oriented matroids belongs to the natural generalization of Zaslavsky's normal class. As such, it has, for arbitrary parameters, expressions obtained by substitutions into (1) computation trees, (2) ported parametrized Tutte polynomials from such trees, (3) various $P$-subbasis expansions, and (4) the ported parametrized corank-nullity polynomial.

We state here the open problem to include ports into the results of Zaslavsky, Bollobas and Riordan, and Ellis-Monaghan and Traldi: Can we classify with universal forms all of the ported parametrized Tutte functions according to their parameters, non-port point values, and the values on oriented or unoriented minors on port ground sets?

5.2 Geometric Lattice Flat Expansion

A formula for the unparametrized ported Tutte (or corank-nullity) polynomials of non-oriented matroids in terms of the lattice of flats (closed sets) and its Mobius function was given in [11]. We generalize: (1) The expansion's monomials $[Q]$ can signify either oriented matroid minors, when $\mathcal{N}$ is oriented, or non-oriented minors when $\mathcal{N}$ is not oriented. (2) The polynomial is parametrized with $r_e, g_e$ for each $e \in E$. The derivation relies on the fact that the oriented or non-oriented matroid minor $[\mathcal{N}/A|P]$ (according to whether $\mathcal{N}$ is oriented or not) depends only on the flat spanned by $A \subseteq E$.  

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Proposition 5.14. Let $\mathcal{N}(P, E)$ be an oriented or unoriented. Let $R_P(\mathcal{N})$ be given from
Definition 4.1. In the formula below, $F$ and $G$ range over the geometric lattice of flats
contained in $E$.

$$R_P(\mathcal{N})(u, v) = \sum_{Q} \sum_{F \in E} u^{\mathcal{N} - \rho Q - \rho F} v^{-\rho F} \prod_{e \in G} (r_e + g_e v)$$

(26)

Proof. It follows the steps for theorem 8 in [11].

Remark: The chirotope values for the oriented matroid minor $\mathcal{N}' = \mathcal{N}/F|P$ are
$\chi_{\mathcal{N}'}(X) = \chi_{\mathcal{N}}(XB_F)$ where $X$ is restricted to sequences over $P$ and $B_F$ is any basis
for the flat spanned by $F$. While this formula defines a chirotope function only up
to a constant sign factor, the oriented matroid (which is what the monomial $[\mathcal{N}/F|P]$
denotes) is uniquely defined. We mention this because when we evaluate the corank-
nullity polynomial to obtain $M_E(\mathcal{N})$ for the unimodular oriented matroid $\mathcal{N}$ substitute
an extensor for each $[\mathcal{N}/A|P]$. However, the object we substitute is $M_0(\mathcal{N}/A|P)$, not
$N/A|P$. It is the unique extensor defined by equation (14) applied to one of the chirotopes
that present $[\mathcal{N}/A|P]$ (or to any other representation of $[\mathcal{N}/A|P]$ for that matter). We
already remarked that equation (14) is unchanged when its argument changes sign!

6 Context and Discussion

6.1 Discrete Laplacian

The combinatorial (or discrete) Laplacian is the matrix of coefficients in the equations
(27) below in variables $\phi_i$, $1 \leq i \leq n$. These discrete Laplace equations model (among
other situations) a resistive electrical network when $\phi_i$ represents the electrical potential
(or voltage) at vertex $i$ and constant $I_i$ represents the current flowing into vertex $i$.

$$\sum_{\{j : e = i j \in E\}} g_{e}(\phi_i - \phi_j) = I_i \quad 1 \leq i \leq n$$

(27)

If each conductance $g_e$ is non-negative, or is either zero or generic, then the rank of
the $n \times n$ Laplacian matrix is $n - k$, where $k$ is the number of connected components in
the $n$ vertex undirected graph whose edges are the $e = i j$ with $g_e \neq 0$. Each order $n - k$
non-singular diagonal submatrix is called a reduced Laplacian. (In §6.2, we will express
Laplace’s equations with an $M_E(N)$ independent of $k$.) The reduced Laplace equations,
together with $\phi_i = 0$ for each vertex $i$ corresponding to a deleted column, model a network
where each such $i$ is contracted into a single grounded vertex whose potential is fixed to
zero and whose external current is unrestricted\(^1\). Since the graph is undirected, equations
(27) imply that the current into the grounded vertex equals the sum of the $I_j$ for the
non-grounded vertices.

\(^1\)Do not confuse with ground set.
The inverse of a reduced Laplacian matrix is called the discrete Green’s function in [18]. This inverse matrix’s elements are each expressed (using Cramer’s rule) by a ratio of an order $n-k-1$ minor to a common order $n-k$ minor denominator. The list of all minors, of all orders, is an example of Plücker coordinates—Here, these are the $\binom{2n}{n}$ maximal minors of the matrix obtained by appending the $n \times n$ identity matrix to the side of the Laplacian.

The Matrix Tree Theorem asserts that each $n-1$ order minor equals $\pm \sum g_T$, the enumeration of spanning trees $T$ by products of edge parameters $g_T = \prod_{e \in T} g_e$. See [8] for similar interpretations of all the minors and for generalizations to directed graphs. The formulas we call “Maxwell’s rules” were given without proof by Maxwell [35], and the dual forms of them were proved by Kirchoff [30]. Maxwell also described the static equilibrium solution for stressed linear elastic framework in terms of enumerations over minimally rigid subframeworks [36]; this enumerated set is the basis set for the rigidity matroid [26]. The one-dimensional case is analogous to the electrical problem. The survey by Biggs [3] covers the discrete Laplacian, the Matrix Tree Theorem, and the use of spanning tree enumeration to solve the discrete Laplace equations, and many additional topics, including the asymmetric discrete Laplacian. Biggs presents the Kirchoff’s solution method and Nerode and Shank [37], also used by Bott and Duffin [6], Smith [39] and Maurer [34]. This method constructs a symmetric projection matrix from a sum of fundamental cocycle matrices, one for each spanning tree. Analysis of basis exchange, i.e., the pivot calculation implies the appropriately weighted matrix sum is symmetric. We plan to present the generalization of this argument to extensors in a future publication.

Tree counting, the discrete Laplacian and electrical network models with parameters have a spectrum of applications including electrical circuit theory, knot theory, random walks and the analysis of Markov chains (see for example [3,17,18,20]). Their application to square dissections is described in [7,45]; Tutte gives a Laplacian based “barycentric embedding” proof of Kuratowski’s Theorem in [44].

It is generally known among electrical engineers in circuit theory that the same kinds of homogeneous rational polynomial functions that appear in Maxwell’s rule occur generally as the coefficients (and minors of them) in all of the linear relationships between the port quantities that define the externally observable characteristics of a linear resistive network. Our results display this principle within the mathematical contexts of the enumerative combinatorics of graphs, oriented matroids and exterior algebra: See Corollary 3.7. Some electrical network analysis software actually enumerates trees and related structures to do “symbolic analysis.” See for example [13,23,41,42].

6.2 Electrical Network Equations and Ports

Consider a graph with two kinds of edges, called ports $P$ and resistors $E$. The graph is directed with an arbitrary edge orientation. Let unimodular extensor $N(P, E)$ present its ported graphic oriented matroid. Let $r$ be the rank of this matroid.

Let $g_e, r_e$ be parameters for each $e \in E$. The extensors $\iota(N)(P_i \cup P_v \cup E)$ and $\nu(N^\perp)(P_i \cup P_v \cup E)$ defined in §3 determine the electrical network equations in the way
expressed by Theorem 2.3 applied to $\mathbb{N}^\perp$.

To be specific, these equations are a linear system on the $|E| + 2|P|$ variables \{\(x_e, \ldots; v_p, \ldots; i_p, \ldots\)} \( \) each \( e \in E \) is associated to variable \( x_e \). Each \( p \in P \) is associated to two variables, \( v_p \) called the voltage and \( i_p \) called the current. Let matrix \( K \) with \( r \) rows be any matrix that presents \( \nu(\mathbb{N}) \). Let \( C \) be any matrix with \(|P| + |E| - r \) rows that presents \( \nu(\mathbb{N}^\perp) \). These matrices express Kirchhoff’s equations combined with a homogeneous expression of Ohm’s law. \( K \) determines the following \textbf{current} equations:

\[
\sum_{p \in P} K_{j,p} i_p + \sum_{e \in E} K_{j,e} x_e = 0 \text{ for } j = 1, \ldots, r.
\]

\( C \) determines the following \textbf{voltage} equations:

\[
\sum_{p \in P} C_{j,p} v_p + \sum_{e \in E} C_{j,e} x_e = 0 \text{ for } j = 1, \ldots, |E| + |P| - r.
\]

It is helpful to see the electrical network equations in terms of \( \mathbb{N} \) directly. Let \( \mathbb{N} \) be any matrix presentation of \( \mathbb{N} \); for example, a reduced oriented incidence matrix of the graph. Let \( \mathbb{N}^\perp \) present \( \mathbb{N}^\perp \); the rows of \( \mathbb{N}^\perp \) comprise a basis for the cycle space of the graph. The current equations can be written:

\[
\sum_{p \in P} N_{j,p} i_p + \sum_{e \in E} N_{j,e} g_e x_e = 0 \text{ for } j = 1, \ldots, r.
\]

The voltage equations can be written:

\[
\sum_{p \in P} N_{j,p}^\perp v_p + \sum_{e \in E} N_{j,e}^\perp r_e x_e = 0 \text{ for } j = 1, \ldots, |E| + |P| - r.
\]

The equations which \( \mathbb{M}_E(\mathbb{N}) \) presents are obtained by eliminating all the variables \( x_e \), \( e \in E \) in the voltage and current equations taken together. Corollary 3.6 indicates the rank of the resulting system of \(|P| \) equations on \( 2|P| \) variables is \(|P| \), provided that the parameters are generic or all positive.

The above analysis illustrates the role for the port element distinction in modeling a physical system. Each non-port element models a completely defined subsystem. The “proto-voltage [39]” \( x_e \) parametrizes the state of one electrical resistor, for example. The behavior of this resistor is thus defined by Ohm’s law: The current is \( g_e x_e \) if and only if the voltage is \( r_e x_e \). The entire model (the graph, for example) specifies all the interactions (via Kirchhoff’s laws, for electricity) between its subsystems. Each port element models an interface pertaining to an interaction of the system with an unspecified environment, for observing the system behavior of interest to the application, and to help specify how certain larger systems are composed out of previously entire subsystems. For us, the environment is assumed, for each port, to constrain the currents into one terminal and out of the other terminal to be equal. Environmental constraints between voltages at terminals belonging to distinct ports are forbidden as well. (Engineering models encompass multi-port elements, whose behavior is specified using multiple port elements. For example, a
linear multiport element is specified a linear constraints among the variables associated with its ports; this generalizes Ohm’s law to so-called multi-terminal resistors. Each of our ported objects can model a single multiport element within a larger model. A topic for future research is to abstract this along the lines given here.)

Let a graph on vertices \{1, \ldots, n\} be given with conductance parameters \(g_e\) for each edge. We now derive Laplace’s equations from the voltage and current equations. Let us append a new vertex 0 (which will be grounded) and \(n\) port edges \(p_i \in P\), with each \(p_i\) directed from vertex 0 to vertex \(i\). For simplicity take parameter \(r_e = 1\) for each edge \(e\) from the original graph. We can choose \(N\) so that the current equations are

\[
i_p = \sum_{e \in E} J_{p,e} g_e x_e, \quad p \in P
\]

and the voltage equations are

\[
x_e = \phi_{p_r} - \phi_{p_s} = \sum_{i=1}^n J_{p_i,e} \phi_{p_i}, \quad \text{where } e = rs, e \in E.
\]

where we used potential (relative to vertex 0) \(\phi_i\) in place of \(v_{p_i}\) and \(J\) is the oriented vertex-edge incidence matrix of the original graph. Laplace’s equation is obtained by eliminating the variables \(x_e\) which represent differences of potential across resistor edges. Indeed, one presentation for \(M_E(N)\) in this case is the \(n \times 2n\) matrix \([I_n \Delta]\) formed by concatenating the identity matrix with the Laplacian matrix \(\Delta\). One manifestation of Theorem 3.3 is therefore that each of the forest enumerating polynomials given by an arbitrary minor, of any order, of the Laplacian is a (non-ported) Tutte function of graphic oriented matroids.

### 6.3 Maxwell’s Rules

One of Maxwell’s rules [35] applies to a pair of ports that do not share a common vertex. See [15] for an elementary derivation. Graph matroid orientation becomes relevant in this situation. Explicit port edges have proven their usefulness in electrical network analysis [16, §13.6]. In this situation, some Plücker coordinates of \(M_E(N)\), as polynomials in \(g_e, r_e\), have terms of opposite sign only when two port edges do not share a vertex. Our contribution is to characterize the signs within the theory of oriented matroids and Tutte functions. It is true that such polynomials can be expressed in terms of minors of the Laplacian; this was done by manipulation of solutions to Laplace’s equation in [7]; see also [45]. However, our extensor and oriented matroid formulation enables the analysis to be done without the introduction of vertices.

Our derivation of Maxwell’s rule for two ports (of which the one port version is a special case) begins with the electrical network equations with \(P = \{p_1, p_2\}\). Let \(M = M_E(N)\) as defined in §3 be as discussed above, and let \(M\) be any \(2 \times 4\) matrix presentation of \(M\). The two equations

\[
M \begin{bmatrix} i_1 \\ i_2 \\ v_1 \\ v_2 \end{bmatrix} = 0
\]
are obtained by eliminating the variables \(x_e, e \in E\) from the electrical network equations. The currents \(i_1, i_2\) in edges \(p_1 = ab\) and \(p_2 = cd\) flow from vertices \(a\) to \(b\), and \(c\) to \(d\) respectively. The voltage (drop) \(v_1\) across edge \(p_1\) is the potential at \(a\) minus the potential at \(b\); the corresponding convention defines the voltage \(v_2\) across edge \(p_2\).

We will assume that all the \(r_e = 1\) and that \(M[p_{1u}, p_{2u}] \neq 0\). The latter is assured from Corollary 3.10 (4.) provided that \(E\) contains a spanning tree and all the \(g_e\) are either positive or generic. Under these conditions, the transfer resistance \(\rho_{21}\) given by \((-v_2)/i_1\) when \(i_2 = 0\) and \(i_1 \neq 0\) is well-defined. (These conventions are used so that when \(p_1\) and \(p_2\) are identical or parallel, \(\rho\) signifies the familiar equivalent resistance which is always positive when \(E\) is connected and all \(g_e > 0\).)

**Proposition 6.1 (Maxwell’s Rule).** Given the electrical network graph model described above, let \(B\) denote the collection of edge sets \(T \subseteq E\) of trees that span the vertex set \(V\), and assume \(\sum_{T \in B} g_T \neq 0\).

For vertices \(i, j, k, l\), let \(B_{ikjl}\) be the collection of all \(F \subseteq E\) for which the subgraph \((V, F)\) is a forest with exactly two trees where vertices \(i\) and \(k\) are in one tree and \(j\) and \(l\) are in the other tree.

The transfer resistance \(\rho_{21}\), where \(p_1 = ab\) and \(p_2 = cd\), is well-defined and is given by:

\[
\rho_{21} = \frac{\sum_{F \in B_{acbd}} g_F - \sum_{F \in B_{bdac}} g_F}{\sum_{T \in B} g_T}. \tag{28}
\]

**Proof.** We will abuse the notation slightly by using \(v_k\) and \(i_k\) for ground set elements \(p_{ku}\) and \(p_{ki}\), and \(v_k, i_k\) for the corresponding extensors, \(k = 1, 2\).

Corollary 3.10 (4.) shows that \(M[v_{1u}v_{2u}] = M[p_{1u}, p_{2u}]\) \(= \sum_{T \in B} g_T \neq 0\), so by Cramer’s rule,

\[
\rho_{21} = -\frac{v_2}{i_1} = -\left( -\frac{M_E(N)[v_{1u}i_{1u}]}{M_E(N)[v_{1u}v_{2u}]}, \right).
\]

Let us apply Corollary 3.8 to the numerator and denominator. To do this, we first calculate \(M_0(N)\) for 6 unimodular extensors \(N(\{p_1, p_2\}, \emptyset)\) that present the 6 oriented matroids on \(P = \{p_1, p_2\}\), which are all graphic. For each of the 6, we can then determine the Plücker coordinate values \(M_0(N)[v_{1u}i_{1u}]\) and \(M_0(N)[v_{1u}v_{2u}]\). Those oriented matroid minors for which one of these values is non-zero will characterize, together with the rank conditions in Corollary 3.8, which forests or trees contribute to each sum. These characterizations of the forest or tree terms, and of their signs, analyzed for each Plücker coordinate, will complete the derivation of Maxwell’s rule.

Four of the 6 oriented matroids are the direct sums of either the loop \(\mathcal{N}_0(p_1)\) or coloop \(\mathcal{N}_1(p_1)\) with either the loop or coloop on \(p_2\). The other two oriented matroids are the orientations of the 2-circuit matroid on \(\{p_1, p_2\}\). Let \(\mathcal{N}_1^+\) denote the oriented circuit \(\pm (+\cdot\cdot\cdot)\); graphically, \(p_1\) and \(p_2\) are parallel. So \(\mathcal{N}_1^-\) denotes the oriented matroid of antiparallel \(p_1\) and \(p_2\). Table 1 lists these six distinct ported oriented matroids \(\mathcal{N}(\{p_1, p_2\}, \emptyset)\), their unimodular extensor presentations \(\mathcal{N}(\{p_1, p_2\})\), and the corresponding extensor values \(M_0(N)\). The \(M_0(N)\) values are easily found up to sign. The signs are given in Proposi-
Table 1: The six oriented matroids on \{p_1, p_2\}.

<table>
<thead>
<tr>
<th>matroid</th>
<th>N</th>
<th>M_0(N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{N}_0(p_1) \oplus \mathcal{N}_0(p_2))</td>
<td>(\pm 1)</td>
<td>(v_1 v_2)</td>
</tr>
<tr>
<td>(\mathcal{N}_0(p_1) \oplus \mathcal{N}_1(p_2))</td>
<td>(\pm p_2)</td>
<td>(i_2 v_1)</td>
</tr>
<tr>
<td>(\mathcal{N}_1(p_1) \oplus \mathcal{N}_0(p_2))</td>
<td>(\pm p_1)</td>
<td>(i_1 v_2)</td>
</tr>
<tr>
<td>(\mathcal{N}_1(p_1) \oplus \mathcal{N}_1(p_2))</td>
<td>(\pm p_1 p_2)</td>
<td>(i_1 i_2)</td>
</tr>
<tr>
<td>(\mathcal{N}^+_1)</td>
<td>(\pm (p_1 - p_2))</td>
<td>((i_1 - i_2)(v_1 + v_2) = i_1 v_1 + i_1 v_2 - i_2 v_1 - i_2 v_2)</td>
</tr>
<tr>
<td>(\mathcal{N}^+_1)</td>
<td>(\pm (p_1 + p_2))</td>
<td>((i_1 + i_2)(v_2 - v_1) = i_1 v_2 - i_1 v_1 + i_2 v_2 - i_2 v_1)</td>
</tr>
</tbody>
</table>

One way to calculate is to analyze the corresponding electrical network with 2 ports and no resistors. For example, the network with oriented matroid \(\mathcal{N}_0(p_1) \oplus \mathcal{N}_1(p_2)\) constrains its port \(p_1\), a loop, to have voltage drop \(v_1 = 0\) but its port \(p_2\), a coloop, to have current \(i_2 = 0\). The current in the loop and voltage across the coloop are unconstrained. The solution subspace corresponds to equations \((v_1 = 0; i_2 = 0)\). The extensors representing these equations are \(\alpha v_1 i_2, \alpha \neq 0\).

Similarly, the network of two parallel ports (case \(\mathcal{N}^+_1\)) constrains the sum of voltage drops going around the oriented circuit to be 0, so Kirchhoff’s voltage law is expressed by \(v_1 - v_2 = 0\). Kirchhoff’s current law in the same network is expressed \(i_1 + i_2 = 0\). Hence the corresponding extensor is \(\pm (v_1 - v_2)(i_1 + i_2)\).

We complete the derivation. First for the denominator. From table 1 the only terms in (22) of Corollary 3.8 that might contribute to \(M_E(N)[v_1 v_2]\) are those for which \(\mathcal{N}/A|P\) is the matroid of two loops \(\mathcal{N}_0(p_1) \oplus \mathcal{N}_0(p_2)\) because the only appearance of \(v_1 v_2\) is in that matroid’s row. The rank conditions further restrict the contributing \(A\) to spanning trees.

Finally, for the numerator \(M_E(N)[v_1 i_1]\), we locate \(\pm v_1 i_1\) in the bottom two rows. These appearances have opposite sign. For \(A \subseteq E\) with \(\mathcal{N}/A|P = \mathcal{N}^-_1\), the contribution is \(-g_A\). The sign is opposite when \(\mathcal{N}/A|P = \mathcal{N}^+_1\), so the distinct orientations of the 2-circuit obtained when contracting \(A\) account for the opposite signs in (28). We can again verify from the rank conditions that the \(F = A\) contributing to the numerator of (28) are the spanning forests with 2 trees containing the indicated vertices as claimed.

Note that the sign dependance of \(M_E(N)\) on \(\epsilon\) and the order of \(P = p_1 p_2\) cancels in the ratio \(\rho_{21}\).

While Theorem 6.1 can be proved by elementary arguments as in [15], the above proof demonstrates how it can be derived from the foregoing theory using algebraic calculations.

Remark: The one port version (§1.3) is immediately derived using a graph where \(p_1\) and \(p_2\) are parallel edges because our proof puts no special conditions on the two ports.
6.4 Ground Set Orientation

The ground set orientation and its role in defining a canonical dual of an extensor, and
our $\mathbf{M}_E(N)$ are motivated by the idea of orientations of orientable manifolds and the
definition of pseudo-forms (or “forms of odd-kind” attributed to de Rham in [24]) in
the mathematics of physics. A pseudo-form is an antisymmetric multilinear operator
$f = f_\epsilon$ that is parametrized by the orientation $\epsilon$ and for which $\epsilon f_\epsilon$ is independent
of the orientation [24]. So, $\epsilon f_\epsilon$ is a well-defined form. In physics, an orientation specifies
one’s convention, say by a right-handed coordinate system, for how one defines a positive
volume or other naturally unsigned physical quantity in terms of an exterior algebra form.
The orientation specifies which ordered bases determine right handed coordinate systems.
In this context, the orientation $\epsilon$ is a $\pm 1$ function for which $\epsilon(B_1)\epsilon(B_2)$ is the sign of the
determinant of the local Jacobian matrix which relates the ordered bases $B_1, B_2$.

6.5 Computational Complexity

Among the non-trivial Tutte invariant functions of succinctly presented graphs or ma-
troids, only two (unless $\mathcal{P} = \#\mathcal{P}$) are polynomial time computable [29, 49]. One such
function, the number of bases, is computable by the Matrix Tree Theorem for graphs
and by its extension (§1.2 (4) with parameters equal to 1) to unimodular matroids.
This number is well-known as the evaluation $T(N, 1, 1)$ of the Tutte polynomial function
$T(N, x, y)$ of matroids $N$. However, computing $T(N, 1, 1)$ is $\#\mathcal{P}$-complete for arbitrary
non-unimodular matroids [48]; this follows because counting the perfect matchings in a
bipartite graph is a $\#\mathcal{P}$-complete problem [47].

The other easy-to-compute invariant is determined by the dimension of the intersection
of a linear subspace and its orthogonal complement over a finite field [49]. The cited
papers prove that all of the other Tutte matroid invariants are either trivial or $\#\mathcal{P}$-
complete. More recently, analogous computational hardness results have been proven
for Tutte functions of graphs (thus implying their hardness for matroids). Among these
results, is that evaluating the parameterized Tutte polynomial for given matroids is a
$\mathbf{VNP}$-complete problem [33]. Here, Valiant’s non-uniform algebraic complexity model
[46] is used, which counts as one deterministic step each evaluation of a polynomial on
constants, variables or previously computed values. $\mathbf{VNP}$ is this model’s class that is
analogous to $\mathbf{NP}$ in the Turing machine model. (See the references in [33]).

A full account of the computational complexity of Tutte invariants of graphs and
matroids is given in [29,38,49–52].

We remark for computationally-inclined readers that:

1. The Tutte equations describe non-unique recursive algorithms to compute Tutte
   functions that generally require $2^{|E|}$ steps.

2. A $|P| \times 2|P|$ matrix representing our extensor can be computed from a graph or
   locally or totally unimodular matrix presentation of a ported oriented unimodular
   matroid.
One suitable algorithm is simple matrix block manipulations followed by Gaussian elimination. Such elimination-based algorithms use polynomial bounded numbers of field operations. Therefore, computation of our extensor generalization of the basis enumerator on graphic and other unimodular matroids is a polynomial time problem when all $r_e = g_e = 1$.

7 Other Directions

It is natural to generalize the current and voltage equations so their respective solutions subspaces (taken to be within $K^S$) are not orthogonal [9]. This leads to the directed graph version of the Matrix Tree Theorem. It did lead as well to a “oriented matroid pair” model for combinatorial conditions for certain equations with monotone non-linearities to be uniquely solvable [12]. These conditions were stated in terms of two oriented matroids with a common ground set having complementary rank and no common non-zero covector; the current paper provides the insight that these two were obtained by deletion/contractions to eliminate port elements. Investigations of a generalization of the Tutte polynomial to two matroids with a common ground set were also begun in [53].

The computation tree formalism was used in greedoid generalizations [25], of the Tutte polynomial because those generalizations do not always have an activities expansions based on element orders. We leave investigation of “ported greedoids” to the future.

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References


