Extensor-Valued Tutte Functions of Regular Oriented Matroids with Parameters and Multiple Distinguished Elements

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abstract

We call an object \( P \)-ported to signify that each \( p \in P \) is distinguished whenever \( p \) is in that object’s ground set \( P \cup E \); \( P \cap E = \emptyset \). So, a \( P \)-ported extensor lies in the exterior algebra of a vector space with a basis \( P \cup E \). We give an extensor valued function \( M_E(N) \) of \( P \)-ported extensors \( N \) and prove it obeys a \( P \)-ported and anti-commutative variation of the parameterized Tutte identities:

\[
M_E(N_1 \cup N_2) = \epsilon(P_1 E_1) \epsilon(P_2 E_2) \epsilon(P_3 P_2 E) M_E(N_1) \vee M_E(N_2)
\]

and

\[
M_E(N) = \epsilon(PE) \epsilon(PE') (g_e M_E(N/e) + r_e M_E'(N \setminus e)) ,
\]

where sets are sequenced, \( e \notin P \), \( E' = E \setminus e \) and the “ground set orientation” \( \epsilon() \) gives sign corrections here and in the definition of \( M_E \). This \( M_E \) restricted to regular (unimodular) extensors \( N \) is shown to be an evaluation of a \( P \)-ported and parameterized (after Zaslavsky) generalization of Crapo’s corank-nullity polynomial generating function \( R_P(N) \) of the oriented matroid \( N \) represented by \( N \). This polynomial is restricted to oriented matroids; it then refines the polynomials given by Las Vergnas on set-pointed matroids by coding orientations of certain minors. When the oriented matroid is graphic and \( P = \emptyset \), \( M_E(N) \) enumerates the spanning trees; \( M_E(N) \) generalizes the Laplacian determinant to an extensor representation of the dimension \( |P| \) linear subspace of all electrical voltage and current values in the port edges feasible under Ohm’s law \( g_e V_e = r_e I_e \) for edges \( e \notin P \) and Kirchhoff’s laws for the entire graph. With the parameters set to 1, the result is a family of \( \binom{3p}{p} \) \( P \)-ported Tutte invariants on those graphs and regular matroids that might include \( p = |P| \) individually

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distinguished and oriented elements. A $p \times 2p$ matrix whose order $p$ minors are this family is easy to compute using linear algebra.

Our technicalities suggest a possible functor from real extensors (with distinguished basis) to realizable oriented matroids, where the categories would include deletion and contraction operators.

1 Introduction

We first introduce Tutte\(^1\) invariants of graphs and matroids, their parametrized (i.e., edge weighted) generalizations called Tutte functions, and the well-known example of the parametrized reduced combinatorial Laplacian matrix’s determinant\(^2\). This determinant is the subject of the famous Tutte Matrix Tree Theorem.

**Definition 1 (Tutte Function [Zas92]).** Given parameters $g_e$ and $r_e$ for each $e \in E$, $F$ is a (strong) Tutte function when it is defined on a minor closed set of matroids and it satisfies the following Tutte Equations.

When $e \in E(\mathcal{N})$ is a non-separating element, i.e., $e$ is neither a loop nor a coloop (i.e., isthmus):

$$F(\mathcal{N}) = g_eF(\mathcal{N}/e) + r_eF(\mathcal{N}\setminus e).$$

When $S(\mathcal{N}_1) \cap S(\mathcal{N}_2) = \emptyset$:

$$F(\mathcal{N}_1 \oplus \mathcal{N}_2) = F(\mathcal{N}_1)F(\mathcal{N}_2).$$

(A weak Tutte function[Zas92] is required to satisfy (1) only.)

The combinatorial (or discrete) Laplacian is the matrix of coefficients in the equations (3) below in variables $\phi_i$, $1 \leq i \leq n$. These discrete Laplace equations model (among other situations) a resistive electrical network when $\phi_i$ represents the electrical potential or voltage at node $i$ and constant $I_i$ represents the current flowing into node $i$ from the environment.

$$\sum_{\{j: e=ij \in E\}} g_e (\phi_i - \phi_j) = I_i \quad 1 \leq i \leq n$$

Assuming each $g_{ij}$ is non-negative, or is either zero or generic, one can prove the rank of the $n \times n$ Laplacian matrix is $n-k$, where $k$ is the number of path-connected components in the $n$ node undirected graph whose edges are the $ij$ with $g_{ij} \neq 0$. Each order $n-k$ non-singular diagonal submatrix is called a reduced Laplacian. The reduced Laplace equations, together with $\phi_i = 0$ for each node $i$ corresponding to a deleted column, model a network where each such $i$ is connected to the “grounded” node whose potential is fixed to zero and whose external current is unrestricted\(^3\). Of course, the original equations demonstrate that the current into the grounded node equals the sum of the $I_j$ for the non-grounded nodes.

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\(^1\)Some authors use the term Tutte-Grothendieck.

\(^2\)This matrix is sometimes called the Kirchhoff matrix.

\(^3\)Do not confuse with ground set.
The inverse of a reduced Laplacian matrix is called the discrete Green’s function in [CY00]. This inverse matrix’s elements are each expressed (using Cramer’s rule) by a ratio of an order \( n - k - 1 \) minor to a common order \( n - k \) minor denominator. The tuple of all minors, of all orders, is an example of Plücker coordinates—Here, these are the \( (n \choose k) \) maximal minors of the matrix obtained by appending the \( n \times n \) identity matrix to the side of the Laplacian.

The Matrix Tree Theorem asserts that each \( n - 1 \) order minor equals \( \pm \sum \text{ products of edge parameters } g_T = \prod_{e \in T} g_e \). More generally, the maximum sized spanning forests of a graph with \( k \) path connected components are enumerated by the order \( n - k \) minors formed by deleting a row and column corresponding to one vertex from each component [Che76]. See [Cha82] for history and similar interpretations of all the minors and for generalizations to directed graphs. It is readily verified that the matroid generalization, the parametrized basis enumerator

\[
F_B(N) = \sum g_B r_B, B \in \mathcal{B}(N),
\]

(4)
satisfies the Tutte equations. Here and elsewhere \( \mathcal{B} = E \setminus B \); \( g_B = \prod \{g_e, e \in B\} \) and \( r_B = \prod \{r_e, e \in \mathcal{B}\} \). Theorem 17, the main result, implies that every member (of which the Laplacian determinant is just one) of the above mentioned tuple of Plücker coordinates satisfies a sign-corrected form of the Tutte equations; indeed the element in exterior algebra with these Plücker coordinates satisfies them when exterior algebra addition and (anticommutative) multiplication replace traditional commutative ring operations.

The recent survey by Biggs [Big97] covers the discrete Laplacian, the Matrix Tree Theorem, and the use of spanning tree enumeration to solve the discrete Laplace equations, and many additional topics, including the asymmetric discrete Laplacian. Biggs presents the solution method dating to Kirchhoff (whose solution actually used cotree enumeration [Kir47]), and Nerode and Shank [NS61], also used by Bott and Duffin [BD53], Smith [Smi72] and Maurer [Mau76]. This method constructs a symmetric projection matrix from a sum of fundamental cocycle matrices, one for each spanning tree. Analysis of basis exchange, i.e., the pivot calculation implies the appropriately weighted matrix sum is symmetric. We plan to present the generalization of this argument to extensors in a future publication.

The corank-nullity polynomial\(^4\) is the universal Tutte invariant obtained from the Tutte polynomial by the change of variables \( u = x - 1 \) and \( v = y - 1 \). But, when its parametrized generalization,

**Definition 2 (Parametrized corank-nullity polynomial).**

\[
R(N, u, v) = \sum_{A \subseteq E} g_A r_A u^{|A| - \rho A} v^{|A| - \rho A},
\]

(5)

was studied, it was found to fail to be universal for Tutte functions. Those Tutte functions that are evaluations of \( R(N, u, v) \) are called normal by Zaslavsky [Zas92].

We state two key definitions:

\(^4\)Some authors call it the rank polynomial.
Definition 3 (P-portered Tutte functions). Make the assumptions from Definition 1, finite set \( P \), and the assumption that for each \( p \in P' \subseteq P \) that is in the ground set \( S(N) = P' \cup E \) \( p \not\in E \).

We say function \( F \) is a (strong) \( P \)-portered Tutte function if \( F \) satisfies the Tutte Equations, with (1) restricted to \( e \not\in P \).

Definition 4 (Parametrized \( P \)-portored corank-nullity polynomial of oriented matroids). Given the above assumptions about \( E, P \) and oriented matroid \( N \) with ground set \( E \cup P \),

\[
R_P(N, u, v) = \sum_{A \subseteq E} [N/A|E] g_A \prod_{u^N - \rho(N/E)} \rho_A, \quad \rho(A) - \rho(A).
\]

where, besides variables \( u, v, [N/P|E] \) signifies the commutative product of symbols (i.e., variables) symbolizing the connected components of the oriented matroid \( N/A|E \) (which denotes oriented matroid minor \( N/A \setminus (E \setminus A) \)).

We will recognize, as a generalization of the Laplacian determinant, and as an anti-commutative variant of a \( P \)-portered Tutte function, and an evaluation of \( R_P(N, u, v) \), the object we construct in this paper: The extensor valued \( P \)-portered function \( M_E(N(N)) = M_E(N) \), where \( \pm N(N) \) is the extensor form of a totally unimodular matrix representation of regular matroid \( N \) with \( S(N) = E \cup P \).

We will face and solve the problems: First, Tutte equation (2) uses a commutative product, but exterior algebra product ("join") is anti-commutative. Second, we must guarantee the sum in our extensor analog of (1) is a decomposable antisymmetric tensor, i.e., really an extensor, since the sum of extensors of step higher than 1 is not necessarily an extensor. Our solution is to make the sign of \( M_E(N) \) depend on the ordering of \( E \) and to arbitrarily specify (using the symbol "\( \epsilon() \)"") which subset orderings are called positive, and then prove a variant of the Tutte equations which has "sign correction factors" dependent on the orderings of the sets appearing in the formulas. For brevity, we continue to use "Tutte" to name such sign-corrected equations and their solutions.

The ground set orientation \( \epsilon() \) and its role in defining a specific dual of an extensor, and our \( M_E(N) \) are motivated by the idea of orientations of orientable manifolds and the definition of pseudo-forms in the mathematics of physics. A pseudo-form is an antisymmetric multilinear operator \( f = f_\epsilon \) that is parametrized by the orientation \( \epsilon \) and it satisfies \( \epsilon f_\epsilon \) is independent of the orientation [Fra01]. (So, \( \epsilon f_\epsilon \) is a well-defined form, independent of orientation \( \epsilon \). In physics, an orientation specifies one's convention, say a "right-handed coordinate system", for defining how one defines a positive volume or other naturally unsigned physical quantity in terms of an exterior algebra form.) In this context, an "orientation" \( \epsilon \) is a \( \pm 1 \) function that satisfies \( \epsilon(B_1) = \sigma(\det J(B_1, B_2)) \epsilon(B_2) \) on all ordered bases where \( J,(.) \) is the Jacobian matrix.

Here and elsewhere, \( \sigma(r) \) denotes the sign of real number \( r \).
1.1 Ports

We introduce physical modeling applications to motivate one reason to distinguish a set $P$ so that the deletion and contraction operations are restricted to $e \notin P$. We call the elements of $P$ ports. Each non-port element models a physical subsystem “wholly part” of the modeled “entire” system: The submodel specifies all its own properties and the entire model specifies all its interactions. Each port element models an interface pertaining to an interaction of the system with an unspecified environment, for observing the system behavior of interest to the application, and to help specify how certain larger systems are composed out of previously entire subsystems\(^5\).

The discrete Laplace equation model (3) has the external node current “constants” $I_i$ and the node voltage variables $\phi_i$ to represent environmental interactions. Now, we first take a view that drops the distinction between so-called independent variables (i.e., constants) and dependent variables, so the model determines the linear subspace (variety) of all the solutions to the equations. (Its dimension is $n - k$.) Second, we recognize that the construction of a non-singular reduced Laplacian matrix defines a model where the node current variables are no longer subject to Kirchhoff current law constraints among themselves, and the node voltages are unique when the non-grounded node currents are specified. This construction makes one node from each path connected component become joined to the common grounded node. The port edges are then constructed by adding one new edge $p_i$ for each node $i \neq 0$, where $p_i$ joins the grounded node (whose potential $\phi_0$ is set to 0) to node $i$. The advantage of port edges is that they simplify the job of defining the oriented matroid $\mathcal{N} = \mathcal{N}(\mathcal{N})$ to which our Tutte function theory will apply. $\mathcal{N}$ is the graphic oriented matroid of the network graph, whose port matroid elements are the port graph edges.

Now that our model is a graph with some edges distinguished as ports, we gain interesting generality by dropping any assumption that the port edges share a common vertex\(^6\). Graphs where all port edges share a common vertex are those that derive from the construction of the reduced Laplacian matrix. However, the general graphs have proven their usefulness in electrical network analysis[CDK87, section 13.6]. We will see new combinatorial phenomena as well: Some values, polynomials in the parameters $g_e, r_e$, of certain Plücker coordinates of the extensor $M_N(N)$ we will construct, will have terms of opposite sign only when two port edges do not share a vertex. This particular phenomenon has been known for a long time[Che76]; our contribution is to characterize it within the theory of oriented matroids and Tutte functions.

In the context of electrical circuits or networks, each non-port graph edge models a resistor, which is a two-terminal device that carries the linear approximation called

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\(^5\)Engineering models encompass multiport elements, whose behavior is specified using multiple port elements. For example, a linear multiport element is specified a linear constraints among the variables associated with its ports; this generalizes Ohm’s law to so-called multi-terminal resistors. A topic for future research is to abstract this along the lines given here. Each of our ported objects can model a single multiport element within a larger model.

\(^6\)We will henceforth replace the term “node” with “vertex.” Node carries the connotation of a distributed conductor at a common potential in some engineering literature
Ohm’s law, which relates the current through the resistor to the voltage across it. Each port edge models one pair of terminal vertices\(^7\). Therefore, each graph edge (abstracted to a matroid element) helps name two model variables: one for current and one for voltage. Our \(M_E(N)\) represents the result after we eliminate all these pairs of variables corresponding the \(e \in E\), starting with a non-singular homogeneous linear equation system that expresses Kirchhoff’s and Ohm’s laws. The Ohm’s law parameter coefficients \(g_e, r_e\) are chosen so \(r_e : g_e\) equals the resistance in edge \(e \in E\).

To be specific, let \(C_I \subseteq \mathbb{R}^{P_I \cup E}\) and \(C_V \subseteq \mathbb{R}^{P_V \cup E}\) be linear subspaces, and define projections and maps:

\[
\begin{align*}
f_I & : \mathbb{R}^{P_I \cup E} \to \mathbb{R}^{P_I \cup E} \text{ so } f_I(p_I) = p_k \text{ and } f_I(e) = e \\
f_V & : \mathbb{R}^{P_V \cup E} \to \mathbb{R}^{P_V \cup E} \text{ so } f_V(p_V) = p_k \text{ and } f_V(e) = e \\
s_I & : \mathbb{R}^{P_I \cup E} \to \mathbb{R}^P \text{ so } s_I(p_I) = p_I \text{ and } s_I(e) = 0 \\
s_V & : \mathbb{R}^{P_V \cup E} \to \mathbb{R}^P \text{ so } s_V(p_V) = p_V \text{ and } s_V(e) = 0 \\
r & : \mathbb{R}^{P_I \cup E} \to \mathbb{R}^E \text{ so } r(p_I) = 0 \text{ and } r(e) = r_e e \\
g & : \mathbb{R}^{P_V \cup E} \to \mathbb{R}^E \text{ so } g(p_V) = 0 \text{ and } g(e) = g_e e.
\end{align*}
\]

where \(p_I, p_V, p_k\) and \(e\) range over \(P_I, P_V, P\) and \(E\) respectively. Assume \(f_I(C_I)\) and \(f_V(C_V)\) are orthogonally complementary subspaces. In the electrical network context (the node-free generalization of discrete Laplacian models) \(f_I(C_I)\) and \(f_V(C_V)\) are respectively the 1-cycle and 1-coboundary (sometimes called cocycle) spaces of a graph. When we say a little more generally “\(N\) represents a regular oriented matroid,” \(f_I(C_I)\) is the row space of a totally unimodular matrix.

\(M_E(N)\) represents some non-singular system of linear equations whose solution set is the projection into \(\mathbb{R}^{P_V \cup P_I}\) by \((s_I, s_V)\) of each solution \((i, v)\) for the equations below:

\[
\begin{align*}
i & \in C_I \text{ “Kirchhoff’s Current Law”} \\
v & \in C_V \text{ “Kirchhoff’s Voltage Law”} \\
gv & = r_i \text{ “Ohm’s Law”}
\end{align*}
\]

The \(P\)-ported Tutte function \(M_E(N)\) is computable using linear algebra in ways similar to the Laplacian determinant. This determinant has a unique status in Tutte function and invariant computation. Evidence reviewed in section 1.4 below indicates little hope for other interesting determinant based or other easy-to-compute Tutte functions or invariants under unrestricted Tutte equations and matroid isomorphism. This is another motivation for extending the theory to \(P\)-ported matroids: The computation of matrix expressions (see [CDK87, section 12.4]) equivalent to \(M_E(N)\) are routine exercises in linear electrical network analysis. See our Section 3 for an example.

The Matrix Tree Theorem can be used to prove the 1 port case of Maxwell’s Rule. In section 7.3 we present a proof of the general 2 port case using the results in this paper.

\(^7\)For us, the environment is assumed to constrain the currents into one terminal and out of the other terminal of each port to be equal; and extra environmental constraints between voltages at terminals belonging to distinct ports are forbidden as well.
Theorem 5 (Maxwell’s Rule). Suppose $\mathcal{N}$ defines a linear resistor network in which for $e \in E$, resistor $e$ has conductance $g_e$, each $r_e = 1$, and the port edge $p_1 \not\in E$ is not a resistor but it demarks the two terminal vertices. Assuming $F_B(\mathcal{N} \setminus p_1) \neq 0$, the equivalent resistance between the two terminal vertices is

$$\frac{F_B(\mathcal{N}/p_1)}{F_B(\mathcal{N} \setminus p_1)}$$

Let us express this in homogeneous coordinates. Let $i_1$ denote the current in port edge $p_1 = \alpha \beta$; physically, this current flows in the environment from the tail vertex $\alpha$ to the head $\beta$, through the network and back to $\alpha$. Let $v_1$ denote the tail-to-head voltage drop defined by $v_1 = \psi_\alpha - \psi_\beta$ where $\psi_i$ is the voltage at vertex $i$. So, for a passive (i.e., power-absorbing) network, $i_1$ and $v_1$ will have opposite sign. Maxwell’s rule now reads

$$F_B(\mathcal{N}/p_1) i_1 + F_B(\mathcal{N} \setminus p_1) v_1 = 0$$

This form is fully general for generic or positive parameters: $F_B(\mathcal{N}/p_1)$ and $F_B(\mathcal{N} \setminus p_1)$ are never both 0 because $\mathcal{N}$ has at least one basis $B$ and either $p_1 \in B$ or $p_1 \not\in B$. The reader is invited to interpret the equation when $\mathcal{N}$ is the loop named $p_1$, the coloop $p_1$, and when endpoints $\alpha, \beta$ are in separate path-connected components of $\mathcal{N} \setminus p_1$, so again $p_1$ is a (matroid) coloop in $\mathcal{N}$. The equivalent resistance is 0 in the first example and is $\infty$ in the other two. Also, if $\mathcal{N}$ is disconnected as a matroid, $F_B(\mathcal{N}/p_1)$ and $F_B(\mathcal{N} \setminus p_1)$ have a common non-zero factor$^8$ which enumerates the bases the direct sum of (matroid) components not containing $p_1$; hence the equivalent resistance is determined by the component containing $p_1$ only.

Our easy-to-compute family of $P$-ported Tutte functions on graphs (depending only on the graph’s oriented matroid) generalizes to arbitrary finite $P = \{p_1, \ldots, p_{|P|}\}$ the coefficient pair $(F_B(\mathcal{N}/p_1), F_B(\mathcal{N} \setminus p_1))$ in the homogeneous form of Maxwell’s rule.

1.2 Extensors

After motivating ports, we introduce into the above context extensors, which are decomposable antisymmetric tensors: They are the standard objects from multilinear algebra for operating with equivalence classes of systems of linear equations and their solution sets.

A finite linearly independent set of $k$ linear equations represents a non-zero step $k$ extensor, i.e., a decomposable $k$-form. It equals the exterior product or exterior algebra join[BBR85, Whi] of $k$ linearly independent vectors or 1-forms. Its Plücker coordinates are the coefficients when the extensor is expanded in terms of exterior products of vectors from a particular basis for the underlying linear space.

The length \(\binom{|P|}{|P|}\) tuple of Plücker coordinates results from generalizing the homogenized Maxwell’s rule to a network with $p = |P|$ ports. Each tuple entry has a different enumeration significance, but those tuple entry functions are not algebraically independent because of the Grassmann-Plücker relations among them. We see the value given by

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$^8$We assume the $g_e : r_e$ are such that the solution is unique. It suffices to assume the network is pathwise connected and these ratios are positive.
our particular Tutte function or invariant on a $P$-ported regular matroid or graph as a single mathematical object, an extensor.

Let $N = N(\mathcal{N})$ be a full row rank regular representation matrix (i.e., it is totally unimodular) of a regular matroid $\mathcal{N}$. The columns are labelled with ground set $P \cup E$. In our notation, $N$ is the exterior product of the vectors corresponding to the rows of matrix $N$. Section 2 covers technicalities which include what we call a ground set orientation, and sign-consistant definitions of oriented matroid minor and dual operations on extensor representations of realizable oriented matroids. We can then properly define extensors $M(N)$ and $M_E(N)$ in section 3 and then state and prove the Tutte identities analogous to (1-2) which $M_E(N)$ satisfies. These definitions and the identities apply to all extensors $N$, but only for totally unimodular $N$ are we sure that $M_E(N)$ is determined by the oriented matroid (a combinatorial structure) $\mathcal{N}$ alone.

We will see that when $\tau_e = \varphi_e = 1$ for all $e \in E$, $M_E(N_1) = M_E(N_2)$ whenever $\mathcal{N} = \mathcal{N}(N_1) = \mathcal{N}(N_2)$ and both $N_i$ are regular representations. Therefore $M_E(\mathcal{N}) = M_E(N_i) = M_E(\mathcal{N})$ is a function defined and invariant on regular $P$-ported oriented matroids $\mathcal{N}$.

1.3 Summary

Here we summarize the results of this paper. The core result is item (3) below; the others define and develop the necessary context.

1. We define a generalization of the inverse of the parametrized reduced combinatorial Laplacian matrix (of a parametrized graph) by an extensor-valued function $M_E$ of $P$-ported regular oriented matroids $\mathcal{N} = \mathcal{N}(N)$ with ground set $E \cup P$ and unimodular extensor representation $N$. For real parameter values, a matrix expression for $M_E$ can be easily computed using linear algebra. The definition also applies to arbitrary $N$ but then the function value no longer depends only on a combinatorial object.

2. Deletion/contraction in Tutte equation (1) corresponds to the elimination of the two variables named by a non-port element in the equations that generalize Kirchhoff’s and Ohm’s laws. Eliminating all the non-port variables, i.e., those named by $E$, determines $M_E(N)$.

3. Our definition of $M_E(N)$ specifies its value within the exterior algebra, not just as an equivalence class of homogeneous coordinates. We seem to need to do this even though the homogeneous coordinates are sufficient to determine the solution subspace, i.e., point in the Grassmannian. The specific value is necessary in order for any equation like (1), which would indicate the sum of two extensors is an extensor (i.e., decomposable antisymmetric tensor), to be valid.

Our definition of $M_E(N)$ depends on an arbitrarily prescribed ground set orientation $\epsilon()$—a nowhere-zero antisymmetric sign function on sequenced ground subsets. Our definition of $M_E(N)$ explicitly uses $\epsilon()$ and our definitions of deletion, contraction and duality of extensors, which all also use $\epsilon()$. These operations represent the
corresponding operations on oriented matroids, except for deletion of oloops and contraction of loops.

The definition and use of a ground set orientation enabled us to give explicit “sign corrections” to help define anti-commutative variants of the Tutte equations that apply to $M_E(\mathcal{N})$. Our proofs are elementary but somewhat intricate.

The Laplacian determinant is the special case when $P = \emptyset$ which implies the step $\rho M_E(\mathcal{N}) = 0$. Our main results signify that not only is the Laplacian determinant a Tutte function: All the minors of it, together as one extensor, a single algebraic object, is an extensor-valued kind of Tutte function. Thus not only can the Laplacian determinant be studied in these terms, but the whole discrete Laplacian operator and its inverse, the discrete Green’s function, be found in that theory.

4. When $|P| \geq 2$, $P$-ported Tutte invariant $M_E(N)$ among others distinguishes some of the orientations of the underlying unoriented regular matroid realized by $N$; particularly, orientations of minors on subsets of $P$. Ordinary Tutte invariants, of course, never distinguish different orientations of the same matroid.

5. The definitions and calculations we detailed in [Cha89], (equivalently, the “big Tutte polynomial of a set pointed matroid” of [Ver99]) are (easily) reinterpreted with oriented matroid minors replacing (non-oriented) matroid minors. The resulting $P$-ported corank-nullity polynomial is therefore universal for $P$-ported Tutte oriented matroid invariants. This polynomial is also easily generalized with the addition of parameters $g_e$ and $r_e$ for each non-port element $e$.

These extensions, to $P$-ported oriented matroids and to non-port element parameters, are (easily) applied to other known formulations of Tutte polynomials and functions which we list here. The original internal/external basis activity value formulation for a graph’s “dichromate” polynomial by Tutte[Tut54] and for matroids by Crapo[Cra68] has been extended to matroid perspectives by LasVergnas[Ver99], Zaslavsky[Zas92] gave a activities based expression for each Tutte function determined by a consistant solution to (1-2). Activities expressions for the “colored Tutte polynomials” in rings, characterized by Bollobas and Riordan[BR99], were reported in a recent note by Traldi[Tra03]. Activities are determined by a particular structure of a computation based on some applications of (1-2). A computation tree-based formalization and generalization of this structure was given[GT90] (and was found to be necessary for greedoid generalizations[GM97]). We will adopt this computation tree approach in our exposition, except we leave investigation of “$P$-ported greedoids” to the future. On the other hand, we also extend from [Cha89] and [Ver99] the use of the Mobius function and enumeration over the matroid’s lattice of flats (closed sets) to express the Tutte polynomial of ported matroids/matroid perspectives.

6. Our $M_E(\mathcal{N})$, defined when $\mathcal{N}$ is regular, is the evaluation of our ported and parametrized corank-nullity polynomial with substitution of $u = v = 0$ and explic-
itly given extensions for the other monomials. It therefore belongs to a generalization of Zaslavsky[Zas92]'s so-called normal class of Tutte functions.

Tree counting, the discrete Laplacian and electrical network models have a spectrum of applications including electrical circuit theory, knot theory, random walks and the analysis of Markov chains (see for example [DS84, Big97, CY99, CY00]). Their application to square dissections is described in [Tut98]; Tutte gives a Laplacian based ("barycentric embedding") proof of Kuratowski's Theorem in [Tut63].

It is generally known among electrical engineers in circuit theory that the same kinds of homogeneous rational polynomial functions that appear in Maxwell's rule occur generally as the coefficients (and minors of them) in all of the linear relationships between the port quantities that define the externally observable characteristics of a linear resistive network. Our results display this principle within the mathematical contexts of the enumerative combinatorics of graphs, oriented matroids and exterior algebra: See Proposition 20. Some electrical network analysis software actually enumerates trees and related structures to generate "symbolic" expressions. See for example [TANW86, TS92, FRVHG98, Cha98]. Maxwell described the static equilibrium solution for stressed linear elastic framework in terms of enumerations over minimally rigid subframeworks[Max65]; this enumerated set is the basis set for the rigidity matroid[GSS93]. The one-dimensional case is analogous to the electrical problem we discussed.

1.4 Computational Complexity

Among the non-trivial Tutte invariant functions of graphs or matroids, only two\(^9\) are polynomial time computable\(^{10}\) [JVVW90, Ver98]. One, the number of bases, is computable by the Matrix Tree Theorem for graphs and by its extension (using a simple Cauchy-Binet theorem based proof) to regular oriented matroids. This number is well-known as the evaluation \(T(N, 1, 1)\) of the Tutte polynomial function \(T(N, x, y)\) of matroids \(N\). Specifically, the number of bases is \(|N N^t|\) where \(N\) denotes here any full row rank totally unimodular matrix realizing \(N' = N(N)\). However, computing \(T(N, 1, 1)\) is \(#\mathcal{P}\)-complete for arbitrary non-regular matroids \(N\).[Ver].

The other easy-to-compute invariant is determined by the dimension of the intersection of a linear subspace and its orthogonal complement over a finite field[Ver98]. The cited papers prove that all of the other Tutte matroid invariants are either trivial or \(#\mathcal{P}\)-complete. More recently, analogous computationally "hardness" results have been proven for Tutte functions of graphs (thus implying their hardness for matroids). Among these results, [LM04] prove that evaluating the parameterized Tutte polynomial for given matroids is a \(\text{VNP}\)-complete problem. Here, Valiant's non-uniform algebraic complexity model[Val79] is used, which counts as one deterministic step each evaluation of a polynomial on constants, variables or previously computed values. \(\text{VNP}\) is this model's class that is analogous to \(\text{NP}\) in the Turing machine model. (See the references in [LM04]).

---

\(^9\)unless \(\mathcal{P} = \#\mathcal{P}\)

\(^{10}\)from the graph's incidence relation
A full account of the computational complexity\textsuperscript{11} of Tutte invariants of graphs and matroids is given in [JWV90, Wel93, Ver98, OW92, Wel95, Wel97].

We remark for computationally-inclined readers that:

1. The Tutte equations describe non-unique recursive algorithms to compute Tutte functions that generally require $2^{\lvert E \rvert}$ steps.

2. A $\lvert P \rvert \times 2\lvert P \rvert$ matrix representing our extensor (i.e., the matrix rows are a basis for a subspace, and the extensor equals the exterior product of step 1 extensors corresponding to each row) can be computed from the input graph or totally unimodular matrix.

One suitable algorithm is simple matrix block manipulations followed by Gaussian elimination. Such elimination-based algorithms use polynomial bounded numbers of field operations. Therefore, computation of our extensor generalization of the basis enumerator on graphic and other regular matroids is a polynomial time problem when all $r_e = g_e = 1$.

3. $P$-ported Tutte polynomials facilitate deriving and expressing “splitting formulas” as in [Neg87, And97, And98, MM03, Mak03, LM04, MM03] for the Tutte polynomial of a matroid when the matroid is a combination of two matroids $\mathcal{N}_1, \mathcal{N}_2$ with $S(\mathcal{N}_1) \cap S(\mathcal{N}_2) = P$ under various combination operations. One of the combinations covered is the modular join. It is the natural matroid abstraction of vertex identifications in graphs and $k$-sums of regular matroids, and it is the subject of [And97]. However, the same framework derives different splitting formulas for other matroid combinations. In [Cha89] we applied this approach to obtain splitting formulas for Edmonds’ matroid union $\mathcal{N}_1 \cup \mathcal{N}_2$ and its dual $(\mathcal{N}_1^\perp \cup \mathcal{N}_2^\perp)^\perp$.

2 Preliminaries

Throughout, the underlying scalar commutative field $\mathbb{R}$ is the reals or the rationals, possibly extended to the polynomial rationals $\mathbb{R}[g_e, r_e]$ in the parameters $g_e, r_e$ when they are indeterminates.

In general, the “input” or given objects for our analyses: parametrized graphs, oriented matroids, extensors, and matrices, will all carry a ground set whose elements occur in two kinds, which we name resistor elements and port elements. We say an object $\mathcal{J}$ is “$P$-ported” when we wish to distinguish each element $p \in P$ as a port element in $\mathcal{J}$ if $p$ is in the ground set of $\mathcal{J}$. The ground set of $\mathcal{J}$ is denoted $S(\mathcal{J})$. $\mathcal{J}$ is strictly $P$-ported means $P \subseteq S(\mathcal{J})$. The notation $\mathcal{J} = \mathcal{J}(P, E)$ signifies that $\mathcal{J}$ is strictly $P$-ported and its “resistor” ground set elements are $E = S(\mathcal{J}) \setminus P$. Letter “$p$” typically denotes a port; “$e$” denotes an element in $E$.

\textsuperscript{11}Unfortunately, the term “graph complexity” which is sometimes used for the number of spanning trees has an entirely different meaning from computational complexity, which quantifies the work a computer must do to calculate the answer to each instance of a parametrized problem. See [GJ79].
Often a mathematical object with one distinguished element within its ground set is called “pointed” as in [Bry71]. Sometimes the object is called “set pointed” or “k-pointed” when there are any number or k distinguished elements. We prefer the term “ported” because of the use of that term in electrical network theory and other engineering fields to signify elements pertaining to an “interface” where a system interacts with its environment, and where a collection of systems whose only common elements are ports would be interconnected.

It is important to emphasize that differently named port elements are distinct. A P-ported isomorphism is an isomorphism \( f \) for which \( f(p) = p \) for every \( p \in P \) in the domain of \( f \), so \( f^{-1}(p) = p \) for \( p \in P \) in the range of \( f \). A P-ported invariant is a function \( F \) for which \( F(f(J)) = F(J) \) for all P-ported objects \( J \) in the domain of \( F \) and for all P-ported isomorphisms \( f \) on them.

Each port element \( p_j \) is associated with two other ground set elements \( i_j = (p_j)_I \) and \( v_j = (p_j)_V \) which we call the port current and voltage elements. Ground set elements \( x_1 \) generate each finite dimensional vector space whose exterior algebra (algebra of antisymmetric tensors under exterior, i.e., antisymmetrized tensor product) we use, (2) label columns of our matrices and (3) occupy the ground sets of our oriented matroids and the edge sets of our graphs.

Strictly speaking, our ground set elements should be considered 1-forms (linear functions) on the vector space of 1-chains. (Physically, this space signifies all assignments of voltage and current in the branches, i.e., the 1-cells of electrical network graphs.) The laws of direct current linear resistive circuit theory, Kirchhoff’s two laws and Ohm’s approximation of linearity of resistance, are homogeneous linear constraints on these assignments. These laws will therefore be represented by subspaces of 1-forms ; i.e., subspaces of the dual of the space of 1-chains. Each union of a linearly independent set of \( k \) linear laws is represented by the exterior product of the corresponding 1-forms. This product is a decomposable \( k \)-form that represents the dimension \( k \) linear subspace.

More specifically: A non-zero step \( r \) extensor \( N \) is an \( r \)-form or other antisymmetric tensor such that \( N \) is decomposable, i.e., \( N \) equals the exterior product \( N_1 \land N_2 \land \ldots \land N_r \) of \( r \) linearly independent step 1 tensors \( N_i = \sum (N_{i,x_j} x_j) \). (We choose the symbol \( \land \) instead of the traditional \( \bigwedge \) to emphasize its geometric significance and to facilitate our future use of \( \wedge \) for the dual operation, following [BBR85, Whi].) The ground subset \( X = \{x_1, \ldots \} \) is a distinguished basis for the vector space \( \mathbb{R}^X \) whose exterior algebra contains \( N \). We use \( V \to V \) to denote mapping from \( \mathbb{R}^X \) to its exterior algebra defined by \( x_i \to x_i \) for \( x_i \in X \), linearity, and \( v_1 \otimes \cdots \otimes v_k \to v_1 \land \cdots \land v_k \). The step 1 tensors are therefore in one-to-one correspondence with the elements of linear space \( \mathbb{R}^X \) under this mapping.

Suppose \( N = N_1, \ldots, N_r \) is a sequence of linearly independent elements of \( \mathbb{R}^X \). Let \( L(N) \subseteq \mathbb{R}^X \) denote the subspace spanned by \( \{N_1, \ldots, N_r\} \). This subspace is represented by the extensor \( N = N_1 \land N_2 \land \cdots \land N_r \). This representation, up to non-zero \( \mathbb{R} \) multiples, is unique and independent of the basis chosen for \( L(N) \). Conversely, given extensor \( N \), the subspace \( L(N) \) it represents is given by \( \{Z \in \mathbb{R}^X | N \land Z = 0 \} \).

We forewarn the reader that elements \( p_j \in P \) and \( e \in E \) will belong to the ground
sets of our “input” objects. But, the extensor value $M_E(N)$ of our Tutte-like function to be defined will have $P_I \cup P_V$ as its ground set. $P_I$ consists of one element $i_j = (p_j)_I$ for each $p_j \in P$, and $P_V$ consists similarly of the corresponding elements $v_j = (p_j)_V$.

Concretely, if we identify each ground set element $x_j$ with the $j$th unit row vector for $j = 1, \ldots, |X|$, $L(N)$ is generated by the rows of $r \times |X|$ matrix $N_{xj}$.

A length $k$ sequence $X$ of ground set elements therefore corresponds to the extensor $X$ by $x_1 \cdots x_k \rightarrow x_1 \otimes \cdots \otimes x_k \rightarrow x_1 \lor \cdots \lor x_k = x_1 \cdots x_k$. In these terms, every antisymmetric tensor $N$ of step $k$ can be expressed by

$$
\sum_{X \subseteq S(N)} N[X]X
$$

where each $N[X] \in \mathbb{R}$ and the only non-zero terms satisfy $|X| = k$. A particular arbitrary ordering is assumed for each $X$, but the term $N[X]X$ is invariant under changes in this ordering, since $N[\cdots]$ and exterior product are both antisymmetric functions. When $N$ is a step $k$ extensor, the $\binom{|S(N)|}{k}$-tuple of $N[X]$ values is called the Plücker coordinates. Their homogeneous equivalence class characterizes the subspace $L(N)$. So these classes parametrize the collection of all dimension $k$ subspaces over a fixed vector space. That collection, considered to be a manifold, is known as a Grassmannian.

A port element models a terminal pair of vertices at which the electrical network interacts with its external environment. (We assume equal and opposite current flow through the two terminals. We also assume no external constraint between the electrical potentials at the terminals of different port edges.) We therefore need the two separate operators $p_{jI} = i_j$ and $p_{jV} = v_j$ for the current and voltage respectively at port element $p_j$.

Each resistor element $e$ is associated with two parameters $g_e$ and $r_e$. In the electrical application, $e$ is a 1-form and the $g$’s and $r$’s are instantiated to constants. When applied to the 1-chain modelling the state of the physical electrical network, $g_e e$ evaluates to the current and $r_e e$ evaluates to the voltage drop in the corresponding resistor. This idea of “proconductance” and “proresistance” parameters to express resistor current and voltage appears in [Smi72].

When the $g_e, r_e$ values are strictly positive integers, the solution is easily related to that of the network in which $e$ is replaced by the series connection of $r_e$ many disjoint parallel connections, each parallel connection is formed of $g_e$ many edges. (The electrical resistance between opposite sides of a uniform conducting rectangle depends only on the material’s constant resistivity and the ratio of its length to width. This fact and its relationship to Lehman’s length/width inequality for graphs is explored by [Cha87, DS84, Lov01].)

We now summarize some notation and recall basics about oriented matroids represented by linear subspaces. Every real linear subspace in $\mathbb{R}^X$ and so every real non-zero extensor $N$ with ground set $X$ represents an oriented matroid we denote by $N = N(N)$, and vise-versa. Specifically,

**Definition 6.** Given either (1) a (possibly 0) real linear subspace $L \subseteq \mathbb{R}^X$, or (2) a non-zero real extensor $N$ with ground set $X$, or (3) a realizable oriented matroid’s covector set $L \subseteq \{+1, -1, 0\}^X$, or else (4) a realizable oriented matroid’s chirotope function
mapping sequences from $X$ into $\{+1, -1, 0\}$, for each alternative there are the other three determining the same oriented matroid in the following ways:

- $\mathbf{N}$ is (up to a non-zero multiple) the exterior product of the vectors comprising any basis for $L$ taken as step 1 extensors over basis $X$. Conversely, $L = L(\mathbf{N})$ denotes the subspace spanned by the factors in any decomposition of $\mathbf{N}$ into an exterior product of step 1 extensors.

- The covectors $\mathcal{L}$ of the oriented matroid represented by $L = L(\mathbf{N})$ are determined by $L$ via\(^\text{12}\):

$$
\mathcal{L}(\mathbf{N}) = \{c \in \{+1, -1, 0\}^X | (c_x)_{x \in X} = (\sigma(l_x))_{x \in X} \text{ for some } l \in L(\mathbf{N})\}
$$

- The Plücker coordinates $\mathbf{N}[B]$ of $\mathbf{N}$ are related to the chiotope $\chi = \chi(\mathcal{L}(\mathbf{N}))$ via

$$
\chi(B) = \sigma(\mathbf{N}[B]) \text{ for each sequence } B
$$

$\chi(B)$ is non-zero only for sequences $B$ with length $|B|$ equal to step of extensor $\mathbf{N}$, which equals the dimension of subspace $L$ and the matroid rank of $\mathcal{L}$.

We will therefore use the same notation $\rho(\mathbf{N})$ for the step of extensor $\mathbf{N}$ as we do for the rank of a matroid.

This summarizes some of our notation. Equivalent defining structures of oriented matroids such as covectors and chirotopes, their axioms, and the proofs that realized oriented matroids are related to linear subspaces as above, and other related elementary topics can be found in expositions of oriented matroids such as [BVS+99, BK92].

We will use matroid terminology such as basis, independent set, loop, coloop, etc. in our work with extensors. For example, $B$ is a basis iff $\mathbf{N}[B] \neq 0$ and $e$ is coloop if $e \in B$ for all bases $B$. We specify a ground set for every extensor, so we can define a dual, and so a loop is a ground set element that does not occur in any basis. We define deletion and contraction operations on extensors: (Note that a rather arbitrary choice for the sign of $\mathbf{N}/e$ is fixed by this definition.)

**Definition 7.** The extensor $\mathbf{N}\setminus e$ has the Plücker coordinates of $\mathbf{N}$ restricted to sequences from $S(\mathbf{N})\setminus e$. The extensor $\mathbf{N}/e$ has Plücker coordinates given by $\mathbf{N}/e[X] = \mathbf{N}[Xe]$.

Thus if we delete a coloop or contract a loop the result is the (zero) extensor 0, which is the additive identity in exterior algebra. It doesn’t represent a matroid. Rank 0 matroids, which are direct products of loops, are represented by the arbitrary non-zero multiples of 1, a step 0 extensor, which is the multiplicative identity in exterior algebra. The ground set of an extensor $\alpha \cdot 1 \neq 0$ is comprised of zero or more loops. (The empty matroid is the rank 0 matroid with empty ground set. It has extensor representation 1 with empty ground

\(^{12}\sigma : \mathbb{R} \to \{+,-,0\} \text{ denotes the sign function.}\)
set. Observe that the empty matroid has non-empty basis collection \( \{0\} \); the Plücker coordinate \( 1[0] = 1 \). We forewarn the reader that these conventions lead to unusual effects of contracting a loop and deleting a coloop, but these definitions are the natural ones on extensor representations of realizable matroids. To formalize, one might extend the category of matroids to contain a zero point to correspond with the zero extensor. The elementary solution is to leave loop contraction and coloop deletion undefined.

Since our results relate exterior algebra addition with oriented matroid deletion and contraction we must face a troublesome issue: The oriented matroid’s chirotope is defined only up to a common sign factor. However, extensors that differ by a non-zero scalar (field \( \mathbb{R} \) multiple are different even though they represent the same point in the Grassmannian, i.e., they define the same linear subspace. We now describe the notions and conventions we propose to resolve this issue.

The symbol \( X \) for a set also denotes an arbitrary ordering of \( X \)'s elements in the sequence \( X = x_1 \ldots x_{|X|} \); this ordering is the same for all occurrences of the same symbol in the same formula. We will also only write formulas such as (9) that are invariant under changes in the ordering denoted by each symbol. (We find it useful to check this condition after each calculation.)

**Definition 8 (Ground set orientation).** An orientation of the ground set \( \epsilon \) is an antisymmetric function into \( \{+1, -1, 0\} \) of sequences of ground set elements that is non-zero on sequences of distinct elements, and satisfies \( \epsilon(\emptyset) = 1 \).

One family of ground set orientations is derived from a fixed linear order on all possible ground set elements using the rule that \( \epsilon(X) = (-1)^v \) where \( v \) is the number of inversions in \( X \) (where an inversion is \((i,j)\) with \( i < j \) and \( x_i > x_j \)). A permutation \( \sigma \in \mathcal{P}_n \) of \( \{1, \ldots, n\} \) is always considered a sequence \( \sigma_1 \sigma_2 \ldots \sigma_n \) with ground set orientation derived from the usual ordering of the integers. However, ground set orientations of matroid elements or graph edges will **not** be assumed to derive from a linear order.

Given length \( n \) sequence \( X = x_1 \ldots x_n \) and \( \sigma \in \mathcal{P}_n \), let \( X_\sigma \) denote \( x_{\sigma_1} \ldots x_{\sigma_n} \). The following routine facts will be used often in our proofs: Of course, \( F \) is antisymmetric means \( F(X_\sigma) = \epsilon(\sigma) F(X) \) for all sequences \( X \) and \( \sigma \in \mathcal{P}_{|X|} \).

**Lemma 9.** Suppose \( \epsilon_1 \) and \( \epsilon_2 \) are arbitrary antisymmetric functions of sequences.

1. If \( n = |X| = |Y|, \sigma \in \mathcal{P}_n, \) and \( A,X,B,Y,D \) are arbitrary sequences of distinct elements, then

   \[
   \epsilon_1(AXB)e_2(CYD) = \epsilon(\sigma)\epsilon_1(AX_\sigma B)e_2(CY_\sigma D) = \epsilon(\sigma)\epsilon_1(AXB)e_2(CY_\sigma D) = \epsilon_1(AX_\sigma B)e_2(CY_\sigma D)
   \]

2. Given ground set orientation \( \epsilon \rightarrow \{+1, -1\}, \) concatenation with a fixed sequence \( Z \) gives us an induced ground set orientation on sequences \( X \) disjoint from \( Z \): \( \epsilon^*(X) = \epsilon(X, Z) \) (since \( \epsilon^* \) is antisymmetric).

PAPER IN PREPARATION (2004), #R00

15
3. For the concatenation denoted \(X, Y\) or \(XY\) of elementwise disjoint sequences, 
\[\epsilon_i(XY) = (-1)^{|Y|} \epsilon_i(YX)\].

When \(N\) is a matrix whose columns are indexed by ground set elements and \(X\) is a finite ordered set of column indexes, \(N(X)\) denotes submatrix of \(N\) with columns \(X\). For \(x \in X\), \(N(x)\) denotes column \(x\) of \(N\) and \(N(x)_v\) denotes the \(v\)th entry in this column. When \(N\) has full row rank \(r\) and \(|X| = r\), \(N[X]\) denotes \(\det N(X)\). A rank \(r\) matrix with \(r\) rows is unimodular when \(N[X] \in \{0, +1, -1\}\) for all \(|X| = r\).

With fixed ground set orientation \(\epsilon\) in hand, we define: (Again, a rather arbitrary sign choice is fixed.)

**Definition 10 (Canonical Dual).** \(N^\perp[X] = N^\perp_{\epsilon}[X] = N[\overline{X}]\epsilon(\overline{X}, X)\) The symbol \(\perp_{\epsilon}\) will be abbreviated by \(\perp\) when \(\epsilon\) is irrelevant or doesn't require emphasis.

Björner et. al. in [BVS+99] observe that the similar expression for a chirotope of the oriented matroid dual of the oriented matroid with a given chirotope is independent of the order chosen \(\overline{X}\) and it is antisymmetric in \(X\).

Our conventions imply that \(A\) in \(A \subseteq X\) denotes an arbitrarily sequenced subset of \(X\). The theorem below gives the classical expression, using our conventions, for the components of an exterior product of antisymmetric tensors in terms the components of the factors. Given a basis for \(\mathbb{R}^X\) and the resulting basis for the exterior algebra, it enables one to express the Plücker coordinates of an exterior product \(N_1 N_2\) in terms of the Plücker coordinates of \(N_1, N_2\).

**Theorem 11 (Laplace’s Expansion).** Given extensors \(N_1, N_2\) and ground set orientation \(\epsilon\):
\[N_1 N_2[X] = \epsilon(X) \sum_{A \subseteq X} N_1[A] N_2[\overline{A}] \epsilon(A, \overline{A})\]  
(11)

where \(\overline{A} = X \ \setminus \ A\).

The only non-zero terms satisfy \(|A| = \rho N_1\) and \(|X| = \rho N_1 N_2\). Note that the scalar \(N_1 N_2[X]\) is independent of the ground set orientation \(\epsilon\). However, the exterior product order is significant; formula (11) demonstrates \(N_1 N_2[X] = N_2 N_1[X](-1)^{\rho(N_1)\rho(N_2)}\). It is also worth observing that (11) is antisymmetric in the order of \(X\).

**Lemma 12.** If \(N \neq 0\) then \(\epsilon(E) NN^\perp[E] > 0\) and if \(N\) is regular then this value is the number of bases \(|B(N(N))|\). (Remember \(\perp = \perp_{\epsilon}\) depends on \(\epsilon\).)

**Proof.** Combine Definition 10 with Theorem 11. \(\square\)

Given a directed (multi)graph \(N\) with edges labelled by elements, let \(N^0\) be the signed incidence matrix, so \(N^0(e)_i = 1\), \(N^0(e)_j = -1\) and \(N^0(e)_k = 0\) for \(k \notin \{i, j\}\) when edge \(e = (i, j)\) is directed from **tail** vertex \(i\) to **head** vertex \(j\). A totally unimodular

\[\text{This formula is used to to define or prove the existence and uniqueness of exterior product, see [HP47].}\]
representation matrix for the oriented graphic matroid of \( \mathcal{N} \) is any full rank row submatrix \( N \) of \( N^0 \). Other such unimodular representation matrices are obtained by left multiplying \( N \) by non-singular square totally unimodular matrices. \( N \) is often called the “reduced ±1 incidence matrix.”

3 A Tutte-Like Extensor Function

Given a \( P \)-ported extensor \( N(P, E) \), we will define a parametrized antisymmetric tensor \( M_E(N) \) by a formula for each Plücker coordinate over ground set (i.e., basis) \( P_I \cup P_V \). An explicit matrix formula for \( M_E(N) \) demonstrates that \( M_E(N) \) is decomposable, so it can be called an extensor. We will then give parametrized identities satisfied by the function \( N(P, E) \rightarrow M_E(N) \) which are analogous to the parametrized \( P \)-ported Tutte equations. Our identities however apply to extensors rather than to commutative ring value. The identities include *sign-correction factors* that depend on the particular ground set orientation \( e \) that was used to define \( M_E(N) \). Thus, after the identity of Theorem 17 part 2 is proved, the definition of \( M_E(N) \) and Proposition 15 below demonstrate that the exterior sum in part 2 is always an extensor.

The definition of \( M_E(N) \) below applies to all extensors \( N \) over \( \mathbb{R}^{E \cup P} \). A special case will be used to define \( M_E(\mathcal{N}) \) when \( \mathcal{N} \) is a regular matroid with \( S(\mathcal{N}) = P \cup E \) and \( \pm N \) is the extensor defined by the rows of any full row rank totally unimodular representation matrix for \( \mathcal{N} \).

Our formulation and notation for operations and properties on extensors are chosen to be similar to the corresponding oriented matroid expressions applied to the oriented matroids represented by the extensors. However, the results in this section belong strictly to exterior algebra.

**Definition 13.** Let \( N = v_1 \lor \cdots \lor v_k \) be a step \( k \) extensor over \( \mathbb{R}^S \). Let \( f : \mathbb{R}^S \rightarrow \mathbb{R}^{S'} \) be a linear map. Then \( N^f \) denotes the extensor over \( \mathbb{R}^{S'} \) equal to \( f(v_1) \lor \cdots \lor f(v_k) \).

When \( S = E \cup P \), let \( S' = E \cup P_I \cup P_V \). When \( N \) is an extensor over \( \mathbb{R}^S \),

\[
N^{P \leftarrow P_I}_{E \leftarrow gE} = N^f
\]

with \( f(p) = p_I \) for each \( p \in P \) and \( f(e) = g_e e \) for each \( e \in E \); and

\[
N^{P \leftarrow P_V}_{E \leftarrow rE} = N^h
\]

with \( h(p) = p_V \) for each \( p \in P \) and \( h(e) = r_e e \) for each \( e \in E \).

In terms of matrices, \( E \leftarrow gE \) signifies multiplying column labelled \( e \) by \( g_e \) for each \( e \in E \). Likewise, \( P \leftarrow P_I \) (\( P_V \)) signifies relabelling column \( p \) by \( p_I \) (\( p_V \)) respectively for each \( p \in P \).

**Definition 14 (An Extensor Valued Tutte Function).** Given

1. extensor \( N \) over \( \mathbb{R}^{P \cup E} \), \( P \cap E = \emptyset \),
2. the ground set orientation $\epsilon$ which defines the canonical dual operator $\perp$,

3. $\epsilon$ preserving port-element-to-port-quantity operators $V : P \rightarrow \mathcal{P}_V$ and $I : P \rightarrow \mathcal{P}_I$;
   $X_I$ and $X_V$ denote the images of $X \subseteq P$ under the operators $I$ and $V$, and

4. resistor element parameters $g_e$ and $r_e$ for each $e \in E$,

let $I \subseteq P$ and $V \subseteq P$ be sequences with $|I| + |V| = |P|$. The Plücker coordinate with index
$I_V$ with respect to the basis $P_I \cup P_V$ of the anti-symmetric tensor $\mathbf{M}_E(N) = \mathbf{M}_E^{I,g,e}(N)$
is defined by

$$\mathbf{M}_E(N)[I_IV] = (N_{E \leftarrow gE} \vee N_{E \leftarrow rE})[I_IV,E]$$  \hspace{1cm} (12)

We can express (12) in matrix terms. Let $p = |P|$ and $e = |E|$. Let $N = N(PE)$
denotes some matrix so product of the rows of
$N(PE)(p_1, \ldots, p_p, e_1, \ldots, e_e)^t = N(PE)(PE)^t$ equals the extensor $\mathbf{N}$. Let $N^\perp$ likewise
denote a matrix representation of extensor $\mathbf{N}^\perp$.

**Example.** We show the matrix form for one regular representation $N$ of the graphic
$P$-ported oriented matroid with $P = \{p_1, p_2, p_3\}$ and $E = \{e_1, e_2, e_3, e_4\}$ for the graph
in figure 1. The rows code the star-cutsets from 3 of the vertices, so they comprise a
basis for the 1-coboundary (or cocycle) space. We also express $\mathbf{N}$ as the exterior product
corresponding to the rows of this matrix.

$$N = \begin{bmatrix} p_1 & p_2 & p_3 & e_1 & e_2 & e_3 & e_4 \\ 1 & 0 & +1 & +1 & +1 & 0 & 0 \\ 0 & +1 & -1 & -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & +1 \end{bmatrix} \quad \begin{align*}
(-p_1 + p_3 + e_1 + e_2) \vee \\
(p_2 - p_3 - e_1 + e_3) \vee \\
(-p_2 - e_2 + e_4) \vee
\end{align*}
$$

Next, we write one regular representation $N^\perp$ of the canonical dual. We have checked
that the sign satisfies Definition (10) with $\epsilon()$ chosen so $\epsilon(p_1p_2p_3e_1e_2e_3e_4) = 1$ by verifying
$N^\perp[e_1e_2e_3e_4] = N[p_1p_2p_3] \epsilon(p_1p_2p_3e_1e_2e_3e_4)$.

$$\begin{bmatrix} p_1 & p_2 & p_3 & e_1 & e_2 & e_3 & e_4 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & +1 & 0 & +1 \\ 0 & +1 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & +1 & -1 \end{bmatrix}$$
Other choices of matrices giving the same Plücker coordinates are convenient for our subsequent elimination operations in columns $E$. The two below were obtained by row operations to maximize the number of ± unit vectors in these columns. These will serve for $N, N^\perp$ for the rest of this example.

\[
N = \begin{array}{ccc|ccc}
p_1 & p_2 & p_3 & e_1 & e_2 & e_3 & e_4 \\
-1 & 0 & +1 & +1 & +1 & 0 & 0 \\
0 & +1 & -1 & -1 & 0 & +1 & 0 \\
-1 & -1 & +1 & +1 & 0 & 0 & +1 \\
\end{array}
\]

\[
N^\perp = \begin{array}{ccc|ccc}
p_1 & p_2 & p_3 & e_1 & e_2 & e_3 & e_4 \\
0 & 0 & +1 & -1 & 0 & 0 & 0 \\
+1 & +1 & +1 & 0 & 0 & 0 & +1 \\
0 & +1 & +1 & 0 & -1 & 0 & 0 \\
+1 & 0 & +1 & 0 & 0 & +1 & 0 \\
\end{array}
\]

Let $G = G(E), R = R(E)$ denote the diagonal matrices of parameters $g_e, r_e; e \in E$. Define order $(p + e) \times (2p + e)$ matrix $M(N)(P_1 P_V E)$ by (and see (7) in section 1.1):

\[
\begin{bmatrix}
N(P) & 0 & N(E)G \\
0 & N^\perp(P) & N^\perp(E)R
\end{bmatrix}
\]  

(13)

Example continued.

\[
M(N) = \begin{array}{cccc|ccc|ccc}
 i_1 & i_2 & i_3 & v_1 & v_2 & v_3 & e_1 & e_2 & e_3 & e_4 \\
-1 & 0 & +1 & 0 & 0 & 0 & g_1 & g_2 & 0 & 0 \\
0 & +1 & -1 & 0 & 0 & 0 & -g_1 & 0 & g_3 & 0 \\
-1 & -1 & +1 & 0 & 0 & 0 & g_1 & 0 & 0 & g_4 \\
0 & 0 & 0 & 0 & 0 & +1 & -r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 & +1 & +1 & 0 & 0 & 0 & r_4 \\
0 & 0 & 0 & 0 & +1 & +1 & 0 & -r_2 & 0 & 0 \\
0 & 0 & 0 & +1 & 0 & +1 & 0 & 0 & r_3 & 0
\end{array}
\]

The exterior row product $M(N)(P_1 P_V E)(P_1 P_V E)^i$ can be expressed in terms of the basis of the $\binom{2p+e}{p+e}$ exterior products of size $p + e$ subsets of $P_1 P_V E$, each with a particular order, since this is a basis for the rank $p + e$ exterior algebra over $R^{P_1 P_V E}$. In this expression,

\[
M(N) = (M_E(N))E + \cdots
\]

where the initial term is the only one with factor $E$.

A matrix expression for $M_E(N)$ can be calculated by row operations to eliminate the columns $E$ in (13). Therefore $M_E(N)$ is decomposable, i.e., is an extensor:

**Proposition 15.** The components (12) are Plücker coordinates for an extensor.
Example continued. We calculate $M_E(N)$ by doing ring operations to eliminate all but one non-zero entry in each $E$ column in $M(N)$. The result is:

$$g_1g_2g_3g_4r_1r_2r_3r_4M(N)$$

equals the following extensor in matrix form:

<table>
<thead>
<tr>
<th>$i_1$</th>
<th>$i_2$</th>
<th>$i_3$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-r_1r_2$</td>
<td>0</td>
<td>$r_1r_2$</td>
<td>0</td>
<td>$g_2r_1$</td>
<td>$g_1r_2 + g_2r_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$r_1r_3$</td>
<td>$-r_1r_3$</td>
<td>$-g_3r_1$</td>
<td>0</td>
<td>$-g_1r_3 - g_3r_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-r_1r_4$</td>
<td>$-r_1r_4$</td>
<td>$r_1r_4$</td>
<td>$-g_4r_1$</td>
<td>$-g_4r_1$</td>
<td>$g_1r_4 - g_4r_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$g_1$</td>
<td>$-g_1r_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$g_4r_1$</td>
<td>$g_4r_1$</td>
<td>$g_4r_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$g_4r_1r_4$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$g_2r_1$</td>
<td>$g_2r_1$</td>
<td>0</td>
<td>$-g_2r_1r_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$g_3r_1$</td>
<td>0</td>
<td>$g_3r_1$</td>
<td>0</td>
<td>0</td>
<td>$g_3r_1r_3$</td>
<td>0</td>
</tr>
</tbody>
</table>

After some cancellation, we can read off the answer from the $3 \times 6$ upper left submatrix:

$$+r_1^2M_E(N) = +r_1^2 M_E(N) = \begin{array}{cccc|ccc}
-i_1 & i_2 & i_3 & v_1 & v_2 & v_3 \\
-1 & 0 & 0 & -r_1r_2 & 0 & g_2r_1 & g_1r_2 + g_2r_1 \\
0 & r_1r_3 & -r_1r_3 & 0 & -g_3r_1 & 0 & -g_1r_3 - g_3r_1 \\
-1 & -r_1r_4 & r_1r_4 & -g_4r_1 & -g_4r_1 & g_1r_4 - g_4r_1 \\
\end{array}$$

Remark: Definition 14 indicates immediately that each Plücker coordinate of $M_E(N)$ is a homogeneous polynomial of degree $|E|$ in the $g_e, r_e$. However, this example demonstrates that there sometimes doesn’t exist a matrix expression for $M_E(N)$ all of whose entries are polynomials. The reader can verify that each order 3 minor of the above matrix is a multiple of $r_1^2$.

Example continued: The calculations below will verify two cases of Maxwell’s rule. See Section 7.3. By Cramer’s rule:

$$\rho_{11} = -\frac{M_E(N)[i_1v_2v_3]}{M_E(N)[v_1v_2v_3]}$$

and

$$\rho_{21} = -\frac{M_E(N)[v_1i_1v_3]}{M_E(N)[v_1v_2v_3]}$$

We calculate:

$$M_E(N)[v_1v_2v_3] = g_1g_2g_3g_4 + g_1g_2r_3g_4 + g_1r_2g_3g_4 + r_1g_2g_3g_4$$

$$M_E(N)[i_1v_2v_3] = (g_1r_3 + g_3r_1)(g_2r_4 + g_4r_2)$$

$$M_E(N)[v_1i_1v_3] = -g_1r_2g_3g_4 + r_1g_2g_3r_4$$

Observe $M_E(N)[v_1v_2v_3]$ is the basis enumerator for $N(N) \setminus P$. It is instructive to observe which minors correspond to each signed term in $M_E(N)[v_1v_2v_3]\rho_{21}$:
Proposition 16. For \( \alpha \in \mathbb{R} \), \( \alpha N \) denotes scalar-extensor multiplication. Then,

\[
M_E(\alpha N) = \alpha^2 M_E(N).
\]

Proof. Every term in the Laplace expansion for each instance of (12) for \( \alpha N \) has a factor of \( \alpha^2 \). \( \square \)

In particular, \( M_E(N) = M_E(-N) \). So, if \( N \) represents a regular \( P \)-ported oriented matroid (such as any graphic matroid) \( \mathcal{N} \), i.e., \( N \) is totally unimodular and \( \chi_N(X) = N[X] \) for all \( X \), then \( M_E(N) = M_E(N) \) is an extensor that depends only on the oriented matroid \( \mathcal{N} \) and the parameters \( g, r \) (together with the ground set orientation \( \epsilon \) and the order of \( E \), which only affect \( M_E(N) \) up to sign).

The main theorem below implies that \( M_E \) for regular matroids \( \mathcal{N} \) can also be calculated by recursion on \( P \)-minors.

Theorem 17. Under the assumptions of Definition (14),

1. If \( N_1 \) and \( N_2 \) have pairwise disjoint ground sets \( S_i = P_i \cup E_i \), then

\[
M_E(N_1 \cap N_2)[I_{1V}V_1V_2] = \epsilon(P_1P_2E)\epsilon(P_1E_1)\epsilon(P_2E_2)
M_{E_1}(N_1)[I_{1V}V_1]M_{E_2}(N_2)[I_{2V}V_2];
\]

more succinctly, the extensor

\[
M_E(N_1 \cap N_2) = \epsilon(P_1P_2E)\epsilon(P_1E_1)\epsilon(P_2E_2)M_{E_1}(N_1) \lor M_{E_2}(N_2).
\]

2. For a given \( e \in E \), let \( E' = E \setminus e \). Let \( N/e, N \setminus e \) be the extensors defined by \( N/e[X] = N[X_e] \) and \( N \setminus e[X] = N[X] \) for all \( X \subseteq P \cup E' \). Then,

\[
M_E(N)[I_{1V}V] = \epsilon(PE)\epsilon(PE')
(g, M_{E'}(N/e)[I_{1V}V] + r, M_{E'}(N \setminus e)[I_{1V}V]),
\]

\[
= \epsilon(PE)\epsilon(PE')
(g, M_{E'}(N/e) + r, M_{E'}(N \setminus e))[I_{1V}V].
\]

In particular, the sum of tensors here is decomposable, i.e., is an extensor, for all values of \( g_e \) and \( r_e \). Recall that this requires the same ground set orientation \( \epsilon \) and sequence of \( E' \) be used when defining the two tensors to be summed.

3. If \( E = \emptyset \),

\[
M_E(N)[I_{1V}V] = M[I_{1V}V] = \epsilon(\overline{V}, V)N[I]N[\overline{V}].
\]

Remarks:

1. Proposition (16) with \( \alpha = \pm 1 \) implies \( M_E(N_1N_2) = M_E(N_2N_1) \). We can also verify this from the right hand side of part 1 using Lemma (9.3) and noting the step of extensor \( M_{E_i}(N_i) \) is \( |P_i| \).
2. If $N \neq 0$, one but not both of $N/e$ and $N \setminus e$ will be the 0 extensor if and only if $e$
 is a loop or a coloop in the matroid of $N$. If $N' = 0$ then $N^\perp = 0$ and $M_E(N') = 0$.

We can therefore write part 2 without restricting $e$ to a non-separator.

The proof of part 3 is immediate from (12) using Definition 10. Part 2, up to the sign
 correction, follows immediately from the fact that minor $[I_I V_E]$ of matrix (13) equals a
 linear combination with coefficients $g_e$ and $r_e$ because the column $e$ belongs to this minor
 no matter which $e \in E$ is specified in part 2. Part 1 is immediate except for the sign
 correction. Our proofs with the sign corrections, although elementary, are surprisingly
 technical and so we defer them to a later section.

**Proposition 18.** The set of $M_E(N)$ obtained as the $g_e$, $r_e$ range over $\mathbb{R}$ for each $e \in E$
 represents the points in a projective subspace of a Grassmannian (which consists of all the
 linear subspaces over $\mathbb{R}^{P \cup P_V}$ with dimension $|P|$).

**Proof.** Induction: Use Theorem 17 part 2 for when $|E| > 0$ and part 3 for the the basis. □

**Proposition 19.** Under the above assumptions,

$$
\epsilon(\nabla, V)\epsilon(PE)M_E[I_IV_E] = \epsilon(P) \sum_{A \subseteq E} N[A]N[\nabla A]g_A r_{\nabla}.
$$

The only non-zero terms in this sum are those for which both $A \cup I$ and $A \cup \nabla$ are bases
 in the matroid of $N$.

**Proof.** Repeated (2) and (3) from Theorem 17 and sign cancellations. □

**Proposition 20.** Plücker coordinate $M_E(N)[I_IV_E]$ is a homogeneous polynomial in the
 $g_e$, $r_e$ whose terms are square-free and have degree $\rho N - |I|$ in the $g_e$ and degree $|E| - \rho N + |I| = |E| + |P| - \rho N - |V| = \rho N^\perp - |V|$ in the $r_e$.

**Proof.** Immediate from Proposition 19, matroid duality and the fact $N[X] \neq 0$ only if
 $|X| = \rho N$. □

The following definition and proposition clarify some of the sign behavior resulting from
 the definitions.

**Definition 21.** A function $F = F^\epsilon(X)$ whose value might depend on the ground set
 orientation $\epsilon$ and on the sequence $X$ is said to be

1. **alternating in $X$** if $F^\epsilon(X_\sigma) = \epsilon(\sigma)F^\epsilon(X)$, where $\epsilon(\sigma)$ is the sign of each permuta-
 tion $\sigma$ in some $P_n$; and

2. **alternating in $\epsilon$** if $F^{\epsilon^\sigma}(X) = -F^\epsilon(X)$.

**Proposition 22.** Let $Q \subseteq P_I \cup P_V$ with $|Q| = |P|$.

1. $M_E(\pm N)[Q]$ is constant under sign change of $\pm N$, and is alternating in $E$, $\epsilon$ and
 $Q$. 

PAPER IN PREPARATION (2004), #R00

22
2. $\epsilon(PE)M_E^{\epsilon}(\pm N)[Q]$ is constant under sign change of $\pm N$ and under changes or reorderings of $\epsilon$ or $E$; it is alternating in $P$ and in $Q$.

3. $\epsilon(PE)M_E^{\epsilon}(\pm N)[P_I]$ enumerates the bases of $\mathcal{N}(\mathcal{N}/P)$, assuming $P$ is independent in the matroid $\mathcal{N}(\mathcal{N})$

$$\epsilon(PE)M_E^{\epsilon}(\pm N)[P_I] = \sum_{B \subseteq E} g_{B^r_B}N^2[B \cdot P],$$

4. and $\epsilon(PE)M_E^{\epsilon}(\pm N)[P_V]$ enumerates the bases of $\mathcal{N}(\mathcal{N} \setminus P)$, assuming $P$ is coindependent in $\mathcal{N}(\mathcal{N})$, by

$$\epsilon(PE)M_E^{\epsilon}(\pm N)[P_V] = \sum_{B \subseteq E} g_{B^r_B}N^2[B].$$

4 Invariants and Corank-Nullity Polynomials

Please recall that extensor $\mathcal{N}$ defines an oriented matroid $\mathcal{N}(\mathcal{N})$ by giving one (of the two oppositely signed) chiotope function’s values as the sign of the corresponding Plücker coordinate: $\chi_N(X) = \sigma(N[X])$. In this section, we first shift focus to oriented matroids, not necessarily realizable, and classes of $P$-ported Tutte invariants and function for them. We conclude by categorizing $M_E(\mathcal{N}(\mathcal{N}))$ within them.

The following definitions extend those given by Zaslavsky[Zas92] and Bollobas and Riordan[BR99] to $P$-ported (oriented) matroids:

**Definition 23 (Strong $P$-ported Tutte Function).** A strong $P$-ported Tutte function with parameters $g_e$, $r_e$ for $e \in E$ is a function $F$ from a minor closed class of $P$-ported (oriented) matroids to a ring for which:

1. If $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ is separable,

$$F(\mathcal{N}) = F(\mathcal{N}_1)F(\mathcal{N}_2) \quad (14)$$

2. For each non-separating non-port element $e \notin P$,

$$F(\mathcal{N}(P, E)) = g_eF(\mathcal{N}/e) + r_eF(\mathcal{N} \setminus e) \quad (15)$$

**Definition 24 (Strong $P$-ported Tutte Invariant).** A strong $P$-ported Tutte invariant $F$ is a $P$-ported strong Tutte function with all parameters equal to 1 and which satisfies $F(\mathcal{N}_1) = F(\mathcal{N}_2)$ whenever $\mathcal{N}_1$ is $P$-isomorphic to $\mathcal{N}_2$.

Each order of application of the parametrized Tutte equations (1) and (2), i.e., Tutte decomposition of $\mathcal{N}$, results in a particular value for $F(\mathcal{N})$ in terms of the parameters and the so-called point values[Zas92]: $x_e = F(\mathcal{N}_1(e))$ and $y_e = F(\mathcal{N}_0(e))$ for each $e \in E$ ($N_r(e)$ denotes the rank $r$ matroid with ground set $\{e\}$). Zaslavsky gave examples
of parameter and point value combinations where different decompositions give different results. The solution he gave [Zas92] is to declare that each given choice of parameters and point values \([P, Q]\) defines a maximal minor closed class \(\mathcal{M}[P, Q]\) of matroids in which all decompositions result in the same value.

Bollobas and Riordan [BR99] gave a more general solution that characterized those parameter and point values, either associated to ground set elements or to color classes of them, for which the Tutte equations determine a unique function into a ring rather than only a field. Their characterization specifies an ideal \(I\) such that the Tutte equations are uniquely satisfied by \(F\) if and only if the range of \(F\) is the contained in ring \(\mathbb{R}[g, r, x, y]/I'\), where \(I'\) is an ideal satisfying \(I \subseteq I'\).

The same complications apply to \(P\)-ported (oriented) matroids, with the addition of a choice of value for \(F(\mathcal{Q}(P'))\) for each connected (oriented) matroid minor \(\mathcal{Q}(P')\) with a ground set \(\emptyset \neq P' \subseteq P\) that occurs in the decomposition. Extending the categorizations given by Zaslavsky, and Bollobas and Riordan of Tutte functions in terms of the parameters and point values to include values on these minors, beyond the “normal” category below, is an open problem.

Just as with non-ported matroids, when the parameters and point values are independent of \(e \in E\), which is implied when \(F\) is a \(P\)-ported Tutte invariant, this complication does not occur. All decompositions application lead to the same \(P\)-ported Tutte polynomial expression for \(F\) in terms of its values on the indecomposable (oriented) matroids over subsets of \(P\) and the isomorphic loops or coloops on \(e \notin P\). We can prove this for \(P\)-ported (oriented) matroids by, following Crapo, giving \(R_P\) by Definition 25 below and using induction to prove its universality. Finally we define the \(P\)-ported Tutte polynomial \(T_P(\mathcal{N})(x, y; \ldots \mathcal{Q}_i \ldots)\) from \(R_P\) with the substitution \(u = x - 1, \ v = y - 1\), where \(x, y\) is the (common) indeterminate point value for each coloop, loop respectively. The details given in [Cha89] are easily adapted from non-oriented indecomposable matroids \(\mathcal{Q}_i\) to oriented ones.

**Definition 25 (Parametrized \(P\)-ported Corank-Nullity Polynomial).**

\[
R(\mathcal{N}(P, E)) = \sum_{A \subseteq E} [\mathcal{N}/A|P] g_A \cdot r^{\rho_{\mathcal{N}} - \rho_{\mathcal{N}/A|P} - \rho_A|A| - \rho_A}
\]

In this formula, an oriented matroid \(\mathcal{N}_i\) enclosed in brackets \([\mathcal{N}_i]\) will denote the commutative product denoted “\(*\)” of connected oriented matroids, each with ground set a non-empty subset of \(P\), formed from the connected components of \(\mathcal{N}_i\). We assume the ground set of \(\mathcal{N}_i\) is \(P\). Each non-empty connected oriented matroid enclosed in brackets \([\mathcal{N}_{ij}]\) names a distinct variable; \(\emptyset = 1\). In general, \([\mathcal{N}_{i1} \oplus \cdots \oplus \mathcal{N}_{ic}]\) denotes \([\mathcal{N}_{i1}] \ast \cdots \ast [\mathcal{N}_{ic}]\) for a separable oriented matroid with \(c\) components. The formula therefore defines a polynomial parametrized by \(g_e, r_e\), whose variables are \(u, v\) together with a distinct variable for every connected component of every minor of \(\mathcal{N}\) obtained by contracting some subset \(A \subseteq E\) and deleting the others \(\overline{A} = E \setminus A\).

It is readily verified that \(R(\mathcal{N}(P, E))\) satisfies the conditions for being a \(P\)-ported Tutte function of oriented matroids. The details published in [Cha89] can be immediately
adapted to (1) the introduction of parameters \( g_e, r_e \) (already generalized in [Zas92] from graphs to matroids) and (2) restricting to oriented matroids and declaring \([\mathcal{N}/A_1|P]\) and \([\mathcal{N}/A_2|P]\) to be distinct monomials when they denote different orientations of the same underlying matroid. Besides the parameters, Definition 25 differs from Las Vergnas’ “big Tutte polynomial” definition [Ver99] only in that \([\mathcal{N}/A|P]\) denotes an oriented matroid minor. We state the details without proof:

**Proposition 26.**

1. If \( e \in E \) is neither a loop nor a coloop in \( \mathcal{N}(P, E) \),

\[
R(\mathcal{N}(P, E)) = g_e R(\mathcal{N}/e) + r_e R(\mathcal{N} \setminus e) \tag{16}
\]

2. If \( \mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2 \) is separable,

\[
R(\mathcal{N}) = R(\mathcal{N}_1) * R(\mathcal{N}_2) = R(\mathcal{N}_2) * R(\mathcal{N}_1) \tag{17}
\]

3. \( R(\mathcal{N}_1(e)) = g_e + r_e u \) and \( R(\mathcal{N}_0(e)) = r_e + g_e v \), for the coloop and loop matroids \( \mathcal{N}_1(e) \) and \( \mathcal{N}_0(e) \) on \( E = \{e\} \), \( P = \emptyset \).

4. If \( \mathcal{N}(P, E) \) has \( E = \emptyset \) then

\[
R(\mathcal{N}) = [\mathcal{N}] \tag{18}
\]

For the purposes of studying \( M_N(N) \), we will verify in Theorem 29 below that \( M_N(N) \) is expressed by an evaluation of \( R_P(\mathcal{N}(N)) \). Zaslavsky named as normal those strong Tutte functions that are obtained by evaluating the parametrized corank-nullity polynomial. Definition 28 extends this to \( P \)-ported (oriented) matroids. Unlike non-ported normal Tutte functions which are defined for all matroids, a ported version \( F \) may fail to be defined on matroids with a minor \( Q(P') \) for which \( F(Q(P')) \) is not defined.

We will then leave for future work the classification of possibly new non-normal strong \( P \)-ported Tutte functions in terms of their parameters, point values and values on connected oriented matroids on subsets of \( P \). The analogous classification problem can be posed for parametrized \( P \)-ported general matroids (without orientation).

**Theorem 27.** Suppose there is a \( P \)-minor closed class \( \mathcal{M} \) of \( P \)-ported (oriented) matroids on which the parametrized Tutte equations are asserted together with the conditions \( F(\mathcal{N}_0(\emptyset, \{e\})) = r_e + g_e v \), \( F(\mathcal{N}_1(\emptyset, \{e\})) = g_e + r_e u \) and \( F(\mathcal{N}(P', \emptyset)) = [\mathcal{N}] \) (for generic \( r_e, g_e \) for each \( e \in E \), generic \( u, v \) and generic \( [\mathcal{N}] \) for each \( \mathcal{N} \) and \( P' \subseteq P \)). Then the unique solution is given by \( R(\mathcal{N}(P, E)) \) for all \( \mathcal{N} \in \mathcal{M} \).

**Proof.** Easy induction. \( \square \)

**Definition 28 (Normal \( P \)-ported Tutte Functions).** Every function of \( P \)-ported (oriented) matroids that is obtained by an evaluation of the \( P \)-ported (oriented) matroid parametrized corank-nullity polynomial is called a normal \( P \)-ported Tutte function \( \mathcal{M} \) of \( P \)-ported (oriented) matroids.
4.1 Classifying our Extensor Valued Tutte Function

We can now classify $M_E$, which is defined only on regular $P$-ported oriented matroids.

**Theorem 29.** The extensor-valued function $\epsilon(P, E)M_E(N)$ (which is alternating in $P$ and the ground set orientation $\epsilon$, and is independent of the order of $E$) of $P$-ported regular oriented matroids is the normal $P$-ported Tutte function obtained by the evaluation using the following substitutions in $R_P(N(N))$ restricted to regular matroids $N$: $u = 0$, $v = 0$, and

$$ [N_1] \ldots [N_c] = \epsilon(P_1 \ldots P_c)\epsilon(P_1) \ldots \epsilon(P_c)M_0(N_1) \vee \ldots \vee M_0(N_c) \quad (19) $$

for each oriented matroid $N$ with ground set $P$ and decomposition into $c \geq 0$ many (oriented) matroid connected components $N = N_1 \oplus \ldots \oplus N_c$. Here $N_i$ is any one of the two oppositely signed extensors defined from the rows of totally unimodular full rank representation matrix of regular matroid $N_i$. (Note (19) is independent of the order of components $N_i$ in $N = N_1 \oplus \cdots \oplus N_c$.)

**Proof.** Immediate from Theorem 17 and Proposition 19, after checking that a term is non-zero in Proposition 19 if and only if the exponents of $u$ and $v$ in Definition 25 are both $0$. \hfill $\square$

Here is another way to look at it:

**Theorem 30.** Suppose $N(P, E)$ is an extensor for which $N[X] \in \{+1, -1, 0\}$ for all $X \subseteq PE$, so $N$ represents the regular oriented matroid $N$ with a chirotope given by $\chi(X) = \sigma(N[X])$. Then, given ground set orientation $\epsilon$, $\epsilon(PE)M_E(N)$ is equal to $R(N)$ evaluated as follows:

1. $u = v = 0$,

2. $[N_i(P_i, \emptyset)] = \epsilon(P_i)M_0(N_i)$ for each connected regular oriented matroid $N_i$ with ground set $P_i$ and any regular extensor representation $N_i$ of $N_i$, and

3. the commutative product $*$ of two extensors $M(N_i(P_i)) * M(N_j(P_j))$ is evaluated by

$$ \epsilon(P_iP_j)\epsilon(P_i)\epsilon(P_j)M(N_i(P_i))M(N_j(P_j)), $$

(which equals

$$ \epsilon(P_jP_i)\epsilon(P_i)\epsilon(P_j)M(N_j(P_j))M(N_i(P_i)) $$

because $\rho M(N_k(P_k)) = |P_k|$ and $P_i \cap P_j = \emptyset$, and the sign of $\epsilon(P_iP_j)$ and the exterior product both change by $(-1)^{|P_i||P_j|}$.)

**Proof.** Immediate from the previous theorem. \hfill $\square$

**Corollary 31.** Under the hypotheses of Theorem 30,

$$ \epsilon(PE)M_E(N) = \epsilon(P) \sum_{A \subseteq E} \sum_{\rho A = |A|} M_0(N/A|P)g_A r_A \quad (20) $$

where $\rho N = \rho N/A|P - \rho A = 0$.
5 Activities, Computation Trees and Basis Expansions

The $P$-ported Tutte equations in Definition 23 specify that every time a matroid is decomposed so a $P$-ported Tutte function or invariant value is expressed in terms of values on smaller matroids, deletion and contraction of each port $p \in P$ is avoided. We will see that the basic results known about various decomposition orders and structures can thereby be extended to $P$-ported matroids. In particular, known expressions for the parametrized Tutte polynomial in terms of an element order $O$, basis enumeration and the internal and external activities of elements are generalized by restricting $O$ to orders in which every $p \in P$ is ordered before\footnote{We use the convention that the deleted/contracted element is the last, i.e., greatest element under order $O$ eligible for reduction.} each $e \in E$. The more general classification of elements as internally or externally, either active or passive with respect to each root-to-leaf path of a computation tree for the matroid (or more generally greedoid) decomposition given by [GT90, GM97] is likewise extended to $P$-ported computation trees. In each case, the result is an interval partition of the boolean subset lattice of $E$. Extending the theory to $P$-ported greedoids might be investigated in the future.

McMahon and Gordon define\cite{GM97} the formal “computation tree” to describe each way to compute $F(N)$ from the Tutte equations. Each (Tutte) computation tree determines from $N$ a polynomial in the parameters and point values. When $N$ is in the domain of a strong Tutte function, each of these computation trees determine the same value. Computation trees are a way of giving a basis expansion for the “Tutte Polynomial”, in terms of a more general definition of internal and external activities of elements with respect to a basis. Their concept is more general because it is based on a computation tree rather than an element order $O$.

We continue with adapting these methods to $P$-ported matroids.

**Definition 32.** Let $N(P,E)$ be a $P$-ported matroid. A $P$-subbasis $F \in \mathcal{B}_p(N)$ is an independent set with $F \subseteq E$ (so $F \cap P = \emptyset$) for which $F \cup P$ is a spanning set for $N(P,E)$ (in other words, $F$ spans $N/P$, see [Ver99].)

**Proposition 33.** For every $P$-subbasis $F$ there exists an independent set $Q \subseteq P$ that extends $F$ to a basis $F \cup Q \in \mathcal{B}(N)$. Conversely, if $B \in \mathcal{B}(N)$ then $F = B \cap E = B \setminus P$ is a $P$-subbasis.

**Proof.** Immediate. \qed

**Definition 34 (Activities with respect to a $P$-subbasis and an element ordering $O$).** Let ordering $O$ have every $p \in P$ before every $e \in E$. Let $F$ be a $P$-subbasis. Let $B$ be any basis for $N$ with $F \subseteq B$.

- **Element $e \in F$ is internally active if $e$ is the least element within its principal cocircuit with respect to $B$. Thus, this principal cocircuit contains no ports. The reader can verify this definition is independent of the $B$ chosen to extend $F$. Elements $e \in F$ that are not internally active are called internally inactive.**
Dually, element $e \in E$ with $e \notin F$ is externally active if $e$ is the least element within its principal circuit with respect to $B$. Thus, each externally active element is spanned by $F$. Elements $e \in E \setminus F$ that are not externally active are called externally inactive.

**Definition 35 (Computation Tree, following [GM97]).** A $P$-ported (Tutte) computation tree for $P$-ported matroid $\mathcal{N}$ is a binary tree whose root is labelled by $\mathcal{N}$ and which satisfies:

1. If $\mathcal{N}$ has a non-separating element that is not a member of $P$, then the root has two subtrees, one a computation tree for $\mathcal{N}/e$ and the other a computation tree for $\mathcal{N} \setminus e$, where $e$ is one such non-separating element. The branch to $\mathcal{N}/e$ is labelled with “$e$ contracted” and the other branch is labelled “$e$ deleted”.

2. Otherwise (so every element in $S(\mathcal{N}) \setminus P$ is a loop or a coloop), the root $\mathcal{N}$ is a leaf.

An immediate consequence is

**Proposition 36.** Each leaf of a $P$-ported computation tree for $\mathcal{N}$ is labelled by the direct sum of some minor (oriented if $\mathcal{N}$ is oriented) of $\mathcal{N}$ with ground set $P$ summed with loop and/or coloop matroids with ground sets $\{e\}$ for various distinct $e \in E$ (possibly none).

**Definition 37 (Activities with respect to a computation tree leaf).** Consider a $P$-ported computation tree for $P$-ported matroid $\mathcal{N}$, a given leaf, and the path from the root to this leaf.

- Let each $e \in E$ labelled “contracted” along this path be called **internally passive**.
- Let each coloop $e \in E$ in the leaf’s matroid be called **internally active**.
- Let each $e \in E$ labelled “deleted” along this path be called **externally passive**.
- Let each loop $e \in E$ in the leaf’s matroid be called **externally active**.

**Proposition 38.** Given a leaf of a $P$-ported computation tree for $\mathcal{N}$: The the set of internally active or internally passive elements constitute a $P$-subbasis of $\mathcal{N}$ which we say “belongs to the leaf”. Furthermore, every $P$-subbasis $F$ of $\mathcal{N}$ belongs to a unique leaf.

**Proof.** For the purpose of this proof, let us extend Definition 37 so that, given a computation tree with a given node $i$ labelled by matroid $\mathcal{N}_i$, $e \in E$ is called internally passive when $e$ is labelled “contracted” along the path from root $\mathcal{N}$ to node $i$. Let $IP_i$ denote the set of such internally passive elements.

It is easy to prove by induction on the length of the path from the root to node $i$ that (1) $IP_i \cup S(\mathcal{N}_i)$ spans $\mathcal{N}$ and (2) $IP_i$ is an independent set in $\mathcal{N}$. The proof of (1) uses the fact that “deleted” elements are non-separators. The proof of (2) uses the fact that for each non-separator $f \in \mathcal{N}/IP_i$, $f \cup IP_i$ is independent in $\mathcal{N}$.
These properties applied to a leaf demonstrate the first conclusion, since each $e \in E$ in the leaf must be a separator by Definition 35.

Given $P$-subbasis $F$, we can find the unique leaf as follows: Beginning at the root, descend the tree according to the rule: At each branch node, descend along the edge labelled "e-contracted" if $e \in F$ and along the edge labelled "e-deleted" otherwise (when $e \not\in F$). (This algorithm also operates on arbitrary $F' \subseteq E$.)

The above definitions and properties enable us to conclude

**Proposition 39.** Given element ordering $O$ in which every $p \in P$ is ordered before each $e \not\in P$, suppose we construct the unique $P$-ported computation tree $T$ in which the greatest non-separator $e \in E$ is deleted and contracted in the matroid of each tree node.

The activity of each $e \in E$ relative to ordering $O$ and $P$-subbasis $F \subseteq E$ is the same as the activity of $e$ defined with respect to the leaf belonging to subbasis $F$ in computation tree $T$.

**Definition 40.** Given ordering $O$ of $E$, the computation tree determined by $O$, or more generally, an arbitrary computation tree for (oriented) matroid $N(P,E)$; together with $P$-basis $F \subseteq E$ which determines a unique leaf in that tree,

- $IA(F) \subseteq F$ denotes the set of internally active elements,
- $IP(F) \subseteq F$ denotes the set of internally passive elements,
- $EA(F) \subseteq E \setminus F$ denotes the set of externally active elements, and
- $EP(F) \subseteq E \setminus F$ denotes the set of externally passive elements.
- $ACT(F)$ denotes $IA(F) \cup EA(F)$.

**Proposition 41.** Given a computation tree for $P$-ported matroid $N$, the boolean lattice of subsets of $E = S(N) \setminus P$ is partitioned by the collection of intervals $[IP(F), F \cup EA(F)]$ (note $F \cup EA(F) = IP(F) \cup ACT(F)$) determined from the collection of $P$-subbases $F$, which correspond to the leaves.

*Proof.* Every subset $F' \subseteq E = S(N) \setminus P$ belongs to the unique interval corresponding to the unique leaf found by the tree descending algorithm given at the end of the previous proof. \hfill \Box

Dualizing, we obtain:

**Proposition 42.** Given the data of Proposition 41, the collection of intervals $[EP(F), E \setminus F \cup IA(F)]$ (note $E \setminus F \cup IA(F) = EP(F) \cup ACT(F)$) constitutes another partition of the same boolean lattice.

*Proof.* The dual of the tree descending algorithm is to descend along the edge labelled "e-deleted" if $e \in F$. \hfill \Box
The following generalizes the “Tutte polynomial” expression given in [Zas92] to P-port (oriented) matroids, as well as Theorem 8.1 of [Ver99].

**Definition 43.** Given parameters \( g_e, r_e, \) point values \( x_e, y_e, \) and (oriented) \( P \)-port matroid \( N \) the Tutte polynomial expression in these and the (oriented) matroid variables \( \{Q_i\} \) determined by a computation tree is equal to

\[
\sum_{F \in B_P} [N/F] P |x_{IA(F)} y_{IP(F)} r_{EP(F)}
\]

**Definition 44.** Given (oriented) matroid variable values, parameter values, point values, we say the Tutte polynomial for \( N \) exists if all the Tutte polynomial expressions determined by all the \( P \)-ported computation trees with root \( N \) are equal. The Tutte polynomial is then defined to be the common value.

For example, for arbitrary or indeterminate (oriented) matroid variables and parameters, and point values given by \( x_e = g_e + r_e u \) and \( y_e = r_e + g_e v \), the Tutte polynomial expression exists and it is equal to the parameterized \( P \)-ported corank-nullity polynomial. We verify this by making these substitutions in Definition 43 analyzing the result using the above partitions.

**Proposition 45.** The corank-nullity generating function \( R_P(N) \) is given by the following activities expansion formula:

\[
R_P(N) = \sum_{F \in B_P} [N/F] P \left( \sum_{IP(F) \subseteq K \subseteq F} g_{K \cup (E \setminus F \setminus L)} u^{IP(F) \setminus (F \setminus L)} r_{E \setminus (F \cup K)} u^{E \setminus (F \cup K)} \right)
\]

In other words, since the same corank-nullity polynomial is obtained by the point value substitutions \( x_e = g_e + r_e u \) and \( y_e = r_e + g_e v \) into the Tutte polynomial expressions from arbitrary orderings or computation trees, all such Tutte polynomial expressions give the same value.

**Proof.** Let \( A = K \cup (E \setminus F \setminus L) \) within the above expansion. We can verify \( \overline{A} = E \setminus A = L \cup (F \setminus K) \). For each \( A \subseteq E \) a unique \( P \)-subbasis \( F \), and two tree leaves are determined, one by the tree descending algorithm and the other leaf by the dual algorithm. Thus \( A \) and \( \overline{A} \) respectively belong to intervals within the boolean lattice partitions of Propositions 41 and 42. In particular, \( A \in [IP(F), F \cup EA(F)] \) and \( \overline{A} \in [EP(F), E \setminus (F \cup IA(F))] \). Therefore the terms in the above sum are equal one by one to the terms in Definition 25 of the corank-nullity polynomial.

For the purposes of this paper it was sufficient to recognize that our extensor valued strong \( P \)-ported parametrized Tutte function of regular oriented matroids belongs to
an easy generalization of Zaslavsky’s normal class. As such, it has, for arbitrary parameters, well-defined realizations obtained by substitutions into computation trees, $P$-ported parametrized Tutte polynomials, various $P$-subbasis expansions, and into the $P$-ported parametrized corank-nullity polynomial of regular oriented matroids.

We leave open the task of continuing the work of Zaslavsky, Bollobas and Riordan by classifying $P$-ported parametrized Tutte functions for graphs, non-oriented matroids and oriented matroids according to their parameters, point values, and the function value on minors on subsets of $P$, i.e., our (oriented) matroid variables.

6 Geometric Lattice Expansion

A formula for the non-parametrized $P$-ported Tutte (or corank-nullity) polynomials of non-oriented matroids in terms of the lattice of flats (closed sets) and its Mobius function was given in [Cha89]. Below we state a generalization: (1) The expansion’s monomials $[Q]$ can signify either oriented matroid minors, when $\mathcal{N}$ is oriented, or non-oriented minors when $\mathcal{N}$ is arbitrary. (2) The polynomial is parametrized with $r_e, g_e$ for each $e \in E$. The derivation relies on the fact that the oriented or non-oriented matroid minor $[\mathcal{N}/A|P]$ (according to whether $\mathcal{N}$ is considered oriented or not) depends only on the flat spanned by $A \subseteq E$.

**Proposition 46.** Let $\mathcal{N}$ be a $P$-ported matroid. Minors $[Q]$ of $\mathcal{N}$ in the summation below can be taken to be oriented only when $\mathcal{N}$ is oriented or they can be taken to be non-oriented in any case. $F$ and $G$ are range over the geometric lattice of flats contained in $E = \mathcal{N} \setminus P$. Let $R_P(\mathcal{N})$ be given from Definition 25.

$$R_P(\mathcal{N})(u, v) = \sum_Q [Q] \sum_{F \leq E} u^{\alpha_{\mathcal{N}/F|P}} v^{-\rho_{E-F}^{-\rho_F}} \sum_{G \subseteq F} \mu(G, F) \prod_{e \in G} (r_e + g_e v)$$  \hspace{1cm} (23)

$$[\mathcal{N}/F|P] = [Q]$$

*Proof.* It follows the steps for theorem 8 in [Cha89]. \hfill \Box

While the chirotope formula for oriented matroid minor $\chi_{\mathcal{N}/F|P}(X) = \chi_{\mathcal{N}}(XB_F)$ with $X$ restricted to sequences over $P$, where $B_F$ is any basis for the flat spanned by $F$) defines $\chi_{\mathcal{N}/F|P}(X)$ only up to a constant sign factor, the oriented matroid (which is what the monomial $[\mathcal{N}/F|P]$ denotes) is uniquely defined. We mention this because when we evaluate the corank-nullity polynomial to obtain $M_E(\mathcal{N})$ for the regular oriented matroid $\mathcal{N}(\mathcal{N})$, we substitute an extensor with Plücker coordinates $\pm 1$ or 0, i.e., a chirotope, for each $[\mathcal{N}/A|P]$. However, the object we substitute is $M_0(\mathcal{N}/A|P)$, not $\mathcal{N}/A|P$. It is the unique chirotope defined by equation (12) applied to one of the chirotopes representing $[\mathcal{N}/A|P]$. We already remarked that equation (12) is unchanged when its argument changes sign!
7 Proofs

7.1 Theorem 17, Part 1

Proof. Let’s begin with the left hand side

\[ M_E(N) \{I_{11}V_{1V}I_{2I}V_{2V}\}. \]

By Definition 14, this

\[ = \left( (N_1N_2)^{P_{E-P_E}} (N_1N_2)^{P_{E-r_E}} \right) \{I_{11}V_{1V}I_{2I}V_{2V}E\}. \]

By Theorem 11, and, in each term, writing \( A = A_1 \cup A_2, A_i \subseteq E_i \), it

\[ = \sum_{A \subseteq E} g_{A_1}g_{A_2}r_{A_1A_2}. \]

(24)

Consider the coefficients above one term at a time, and apply Definition 10. The coefficient of \( g_{A_1}g_{A_2}r_{A_1A_2} \) equals

\[ (N_1N_2)\{I_{11}I_{21}A_1A_2\} \cdot (N_1N_2)\{V_1V_2A_1A_2\} \cdot \epsilon(I_{11}I_{21}A_1A_2V_{1V}V_{2V}A_1A_2) \epsilon(I_{11}V_{1V}I_{2I}V_{2V}E). \]

Several applications of Lemma 9, part 1 show this coefficient equals

\[ (N_1N_2)\{I_{11}A_1I_{21}A_2\} \cdot (N_1N_2)\{V_1A_1V_2A_2\} \cdot \epsilon(V_1A_1V_2A_2V_1A_1V_2A_2) \epsilon(I_{11}V_{1V}I_{2I}V_{2V}E). \]

(25)

The index sequence orders written here and the exterior product definition give

\( (N_1N_2)\{X_1X_2\} = N_1\{X_1\}N_2\{X_2\} \) when \( X_i \subseteq S(N_i) \) and \( S(N_1) \cap S(N_2) = \emptyset \). Therefore,

\[ (N_1N_2)\{I_{11}I_{21}A_2V_{1V}V_{2V}A_1A_2\} \epsilon(I_{11}V_{1V}I_{2I}V_{2V}E). \]

Applying Definition 10 again, this equals

\[ N_1\{I_{11}A_1\}N_1\{V_1A_1\}N_2\{I_{21}A_2\}N_2\{V_2A_2\} \epsilon(V_1A_1V_2A_2)\epsilon(V_2A_2V_2A_2). \]

(26)

Hence, from (25) and (26) we see the signs of the coefficients of \( N_1N_2\{I_{11}I_{21}A_2\}(N_1N_2)^{P_{E-r_E}} \{V_1V_2A_1A_2\} \) and \( N_1\{I_{11}A_1\}N_1\{V_1A_1\}N_2\{I_{21}A_2\}N_2\{V_2A_2\} \) differ by
the product of 5 signs:

\[ \epsilon(V_1A_1V_2A_2V_1A_1V_2A_2) \cdot \]
\[ \epsilon(I_{1I}A_1I_{2I}A_2V_1V_1A_1V_2A_2) \cdot \]
\[ \epsilon(V_1A_1V_2A_2V_1A_1V_2A_2) \cdot \]
\[ \epsilon(I_{1I}V_1I_{2I}V_2V_2E) \]

By applying lemma 9 part 1 to the first two \( \epsilon() \), this

\[ = \epsilon(V_1A_1V_1A_1V_2A_2V_2A_2) \cdot \]
\[ \epsilon(I_{1I}A_1V_1V_1A_1I_{2I}A_2V_2A_2) \cdot \]
\[ \epsilon(V_1A_1V_2A_2V_1A_1V_2A_2) \cdot \]
\[ \epsilon(I_{1I}V_1I_{2I}V_2V_2E) \]

By applying the same lemma to the first and third, then the first and fourth \( \epsilon() \)s above, and noting facts like \( P_{1\sigma} = V_1V_1 \) for \( P_2, E, E_1 \) and \( E_2 \), it

\[ = \epsilon(P_1E_1P_2E_2) \cdot \]
\[ \epsilon(I_{1I}A_1V_1V_1A_1I_{2I}A_2V_2A_2) \cdot \]
\[ \epsilon(P_1E_1)\epsilon(P_2E_2) \cdot \]
\[ \epsilon(I_{1I}V_1I_{2I}V_2V_2E) \]

Holding \( A_2 \) (and \( A_1 \)) constant, temporarily define induced ground set orientation \( \epsilon^*(\) by

\[ \epsilon^*(I_{1I}A_1V_1V_1A_1) = \epsilon(I_{1I}A_1V_1V_1A_1, I_{2I}A_2V_2A_2). \]

The sum (over \( A_1 \)) of the terms in (24) with \( A_2 \) constant is seen to be:

\[ g_{A_2}r_{A_2} \cdot \]
\[ \epsilon(P_1E_1P_2E_2)\epsilon(P_1E_1)\epsilon(P_2E_2) \cdot \]
\[ \epsilon(I_{1I}V_1I_{2I}V_2V_2E) \cdot \]
\[ \sum_{A_1 \subseteq E_1} g_{A_1}r_{A_1} \epsilon^*(I_{1I}A_1V_1V_1A_1)N_{2}^{P_{t}-P_{l}}[I_{2I}A_2]N_{1}^{P_{t}-P_{l}}[V_2V_2A_2] \cdot \]
\[ N_{1}^{P_{t}-P_{l}}[I_{1I}A_1]N_{1}^{P_{t}-P_{l}}[V_1V_1A_1] \]

The sum in this expression can now be recognized as a Laplace expansion expressed using the temporary \( \epsilon^*(\); so by Definition 14 this sum

\[ \sum_{A_1 \subseteq E_1} \cdots = \]
\[ \epsilon^*(I_{1I}V_1V_1E_1) \left( N_{1}^{P_{t}-P_{l}}[V_1V_1E_1] \right) [I_{1I}V_1V_1E_1] \]
With the explicit summation over \( A_1 \) eliminated, we now replace \( \epsilon^*() \) by its definition (which depends on \( A_2 \)) and write the entire sum over \( A_2 \subseteq E_2 \), with constant terms factored to the left:

\[
\epsilon(P_1 E_1 P_2 E_2)\epsilon(P_1 E_1)\epsilon(P_2 E_2) \cdot \\
\epsilon(I_{11} V_{1V} I_{21} V_{2V} E) \cdot \\
\left( P_{E} P_{I} N_1^{-E} \vee N_1^{E} P_{I}^{-P_{E}} \right) [I_{11} V_{1V} E_1] \cdot \\
\sum_{A_2 \subseteq E_2} g_{A_2}^{-F_{A_2}} \\
\cdot N_2^{P_I^{-P_I}[I_{21} A_2] N_2^{P_I^{-P_I}}[V_{2V} A_2]} \cdot \\
\epsilon(I_{11} V_{1V} E_1, I_{21} A_2 V_{2V} A_2)
\]

We rewrite this with the definition of \( M_{E_1} \) after recognizing another Laplace expansion for \( M_{E_2}(N_2)[I_{21} V_{2V}] \):

\[
\epsilon(P_1 E_1 P_2 E_2)\epsilon(P_1 E_1)\epsilon(P_2 E_2) \cdot \\
\epsilon(I_{11} V_{1V} I_{21} V_{2V} E) \cdot \\
M_{E_1}[I_{11} V_{1V}] \cdot \\
\epsilon(I_{11} V_{1V} E_1, I_{21} V_{2V} E_2) \cdot \\
M_{E_2}[I_{21} V_{2V}]
\]

We can now use \( |I_{21} V_{2V}| = |P_2| \) and lemma 9 part 2 twice to prove:

\[
\epsilon(P_1 E_1 P_2 E_2)\epsilon(I_{11} V_{1V} E_1, I_{21} V_{2V} E_2) = \epsilon(P_1 P_2 E)\epsilon(I_{11} V_{1V} I_{21} V_{2V} E).
\]

With one final \( \epsilon()\epsilon() \) substitution and one sign cancellation, we can conclude that the sign correction factor is

\[
\epsilon(P_1 P_2 E)\epsilon(P_1 E_1)\epsilon(P_2 E_2)
\]

and the identity of part 1 of Theorem 17 is proven. \( \square \)

### 7.2 Theorem 17, Part 2

**Proof.** The Laplace expansion for the left hand side

\[
M_E(N)[I_{1V} V] = \sum_{X \subseteq E} g_X \epsilon(M_X) N[X] N^{\perp}[X] \epsilon(I_{1V} X_{1V} X) \epsilon(I_{1V} E)
\]

equals by definition of \( \perp \),

\[
\sum_{X \subseteq E} g_X \epsilon(M_X) N[X] N^{\perp}[X] \epsilon(V X, \overline{V X}) \epsilon(I_{1V} X_{1V} X) \epsilon(I_{1V} E).
\]
Take \( E' = E \setminus e \) and let \( A' \subseteq E' \), \( A = A'e \), \( \overline{A} = E \setminus A \) so \( e \not\in \overline{A} \) and \( \overline{A'} = \overline{A}e \). Each \( A' \) contributes up to one non-zero term within

\[
g_{A'}r_{A'} g_e \left( N[A]N[V \overline{A}]\epsilon(V A, V \overline{A})\epsilon(I_1 A' V \overline{A} e)\epsilon(I_1 V e) + r_e N[A']N[V \overline{A'}]\epsilon(V A', V \overline{A}e)\epsilon(I_1 A' V \overline{A}e)\epsilon(I_1 V e) \right).
\]

Clearly, these terms appear up to sign in \( g_e M_{E'}(N/e)[I_1 V e] \) and \( r_e M_{E'}(N \setminus e)[I_1 V e] \) respectively, but we have to derive the sign correction factors. Definition 7 says \((N/e)[X] = N[X e] \) for \( e \notin X \) and \((N/e)^\perp[Y] = (N/e)[Y] \epsilon(Y, Y) \) for \( e \notin Y \) and \( Y = E' \setminus Y \). Therefore, the first term can be expressed:

\[
(g_e(N/e)[I A'](N/e)^\perp[V \overline{A}]\epsilon(V A' V \overline{A})\epsilon(V A, V \overline{A})\epsilon(I_1 A' V \overline{A} e)\epsilon(I_1 V e))
\]

Since \( e \in A \), Lemma 9 part 1 applied to the middle two \( \epsilon() \)s shows the sign factor is

\[
\epsilon(V A' V \overline{A})\epsilon(V A', V \overline{A} e)\epsilon(I_1 A' V \overline{A} e)\epsilon(I_1 V e)
\]

because \( e \in A \). Now apply that lemma to the first two \( \epsilon() \)'s to get

\[
\epsilon(V V A' \overline{A})\epsilon(V V A' \overline{A} e)\epsilon(I_1 A' V \overline{A} e)\epsilon(I_1 V e).
\]

The permutations \( V V \) of \( P \) and \( A' \overline{A} \) of \( E' \) each appear in both of the first two of these new \( \epsilon() \)s, so another expression for the sign is

\[
\epsilon(P E')\epsilon(P E' e)\epsilon(I_1 A' V \overline{A} e)\epsilon(I_1 V e).
\]

We can now gather the two Plücker coordinate factors and the one sign factor \( \epsilon(I_1 A' V \overline{A} e) \) which are the only factors that now depend on \( A' \) to get:

\[
\epsilon(P E')\epsilon(P E' e)\epsilon(I_1 V e) g_e \sum_{A' \subseteq E'} g_{A'}r_{A'}(N/e)[I A'](N/e)^\perp[V \overline{A}]\epsilon(I_1 A' V \overline{A} e).
\]

Next, apply the substitutions \( P \leftarrow P \ldots \) and \( E' \leftarrow h E \ (h = g, r) \) to get:

\[
\epsilon(P E')\epsilon(P E' e)\epsilon(I_1 V e) g_e \sum_{A' \subseteq E'} \epsilon(N/e)[I A'](N/e)^\perp[V \overline{A}]\epsilon(I_1 A' V \overline{A} e).
\]

This equals the Laplace expansion of the following, with the induced ground set orientation \( \epsilon^*(X) = \epsilon(X e) \) used in (11):

\[
\epsilon(P E')\epsilon(P E' e)\epsilon(I_1 V e) \epsilon((N/e)^\perp[V \overline{A}])(I_1 V 
\end{equation}

The sign correction calculated so far for the \( g_e M_{E'}(N/e)[I_1 V e] \) term is

\[
\epsilon(P E')\epsilon(P E' e)\epsilon(I_1 V e)\epsilon(I_1 V e)\epsilon(I_1 V e).
\]

After applying the same permutation \( E' e \) of \( E \) twice, we see this equals

\[
\epsilon(P E')\epsilon(P E)\epsilon(I_1 V e)\epsilon(I_1 V e).
\]

The last two factors cancel and the result is the sign correction that we claimed.

The proof that the same sign correction applies to the \( r_e M_{E'}(N \setminus e)[I_1 V e] \) term uses similar methods and is omitted for the sake of space. □
7.3 Maxwell’s Rule

Let \( p_1 \) and \( p_2 \) be port edges which demark two pairs of “input” and “output” terminals. These terminals belong to a network whose other edges model resistors, where resistor named \( e \) has conductance \( g_e \). Conductance is the reciprocal of resistance. By Ohm’s law, resistor \( e \) constrains the voltage drop \( v_e \) going along \( e \) and the current flow \( i_e \) in the same direction to satisfy \( i_e = g_e v_e \). Assume the resistor edges comprise a path-connected spanning subgraph \(^{15}\), and all the conductance values are positive (modeling physically realistic passive, i.e., power absorbing, linear resistors.)

Suppose an external electrical power source constrains the current flowing out of the tail vertex and into the head vertex of \( p_1 \) to have value \( i_1 \). Then, a well-defined voltage drop \( v_2 = v_{ad} = \phi_c - \phi_d \) can be observed between the terminals demarked by \( p_2 = ad \), where \( \phi_i \) is the voltage at vertex \( i \).

The following is known in electrical network (or circuit) theory as Maxwell’s rule, which asserts that the constant \( \rho_{21} \) in \( v_2 = \rho_{21} i_1 \) is the quotient of two polynomials in the conductance values, and each is an enumerator for a kind of forest.

\textbf{Theorem 47.} Let \( \mathcal{N} \) be a graph model of an electrical network consisting of two oriented port edges \( p_1 = ab \) and \( p_2 = cd \) and some other resistor edges \( E \), each \( e \in E \) having the conductance parameter \( g_e \). \( V \) is the vertex set.

Let \( \mathcal{B} \) denote the collection of edge sets \( T \subseteq E \) of spanning trees of \( \mathcal{N} \), which are forests with one path-connected component, i.e., \textit{tree}. Assume at least one \( T \in \mathcal{B} \) satisfies \( g_e \neq 0 \) for all \( e \in T \).

For vertices \( i, j, k, l \), let \( \mathcal{B}_{ik,ji} \) be the collection \( F \subseteq E \) where the subgraph \( (V, F) \) is forest of with exactly two trees where vertices \( i \) and \( k \) are together in one tree and vertices \( j \) and \( l \) are in the other tree.

\[
\rho_{21} = \frac{\sum_{F \in \mathcal{B}_{ik,ji}} g_F - \sum_{F \in \mathcal{B}_{ik,hd}} g_F}{\sum_{T \in \mathcal{B}} g_T} \quad (29)
\]

Theorem 5 is a corollary.

\textit{Proof.} Let us declare \( i_1, i_2, v_1, v_2 \), signifying the currents and voltages respectively at ports \( p_1 \) and \( p_2 \), to be a basis for \( \mathbb{R}^4 \). Strictly speaking, they are the 1-forms (linear operators) operating on the state space of the network. The solution, i.e., the result of analyzing each (linear resistive) network with these two ports, consists of linear subspace of the constraints on the values of our 1-forms due to the network. This subspace will be represented by a \( k \)-form (extensor) that is the exterior product of linear combinations of the \( i_1, i_2, v_1, v_2 \).

We will first calculate \( M_\emptyset(\ldots) \) for all 6 possible networks with two ports and no resistors, which correspond to the 6 different oriented matroids with ground set \( \{ p_1, p_2 \} \). We will then substitute these results into the expression for \( M_E(\mathcal{N}) \) from Corollary 31.

\(^{15}\)A subgraph of graph \( (V, E) \) is a graph \( (V', E') \) for which vertices \( V' \subseteq V \) and edges \( E' \subseteq E \). A spanning subgraph \( (V, E') \) contains all the vertices.
Four of the above 6 oriented matroids are the direct products of either the loop $\mathcal{N}_0(p_1)$ or coloop $\mathcal{N}_1(p_1)$ named $p_1$ with either the loop or coloop on $p_2$. The other two oriented matroids are the possible orientations of the matroid whose one circuit is $\{p_1, p_2\}$. Let $\mathcal{N}_1^+$ denote the case with oriented circuit $\pm(+-)$; graphically, $p_1$ and $p_2$ are parallel. So $\mathcal{N}_1^-$ denotes the oriented matroid of antiparallel $p_1$ and $p_2$. The table lists these six distinct strictly $\{p_1, p_2\}$-ported oriented matroids $\mathcal{N}(\mathbb{N})$ with $E = \emptyset$, all the regular extensor representations $\mathbb{N}$ over ground set $\{p_1, p_2\}$ for each, and the extensor value $M_0(\mathbb{N})$ of our $P$-ported invariant: The $M_0(\mathbb{N})$ values are easily defined up to sign. The signs are given in Figure 2: The six oriented matroids on $\{p_1, p_2\}$.

<table>
<thead>
<tr>
<th>matroid</th>
<th>$\mathbb{N}$</th>
<th>$M_0(\mathbb{N})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N}_0(p_1) \oplus \mathcal{N}_0(p_2)$</td>
<td>$\pm 1$</td>
<td>$v_1 v_2$</td>
</tr>
<tr>
<td>$\mathcal{N}_0(p_1) \oplus \mathcal{N}_1(p_2)$</td>
<td>$\pm p_2$</td>
<td>$i_2 v_1$</td>
</tr>
<tr>
<td>$\mathcal{N}_1(p_1) \oplus \mathcal{N}_0(p_2)$</td>
<td>$\pm p_1$</td>
<td>$i_1 v_2$</td>
</tr>
<tr>
<td>$\mathcal{N}_1(p_1) \oplus \mathcal{N}_1(p_2)$</td>
<td>$\pm p_1 p_2$</td>
<td>$i_1 i_2$</td>
</tr>
<tr>
<td>$\mathcal{N}_1^-(p_1 - p_2)$</td>
<td>$i_1 - i_2(v_1 + v_2) = i_1 v_1 + i_1 v_2 - i_2 v_1 - i_2 v_2$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{N}_1^+(p_1 + p_2)$</td>
<td>$(i_1 + i_2)(v_2 - v_1) = i_1 v_2 - i_1 v_1 + i_2 v_2 - i_2 v_1$</td>
<td></td>
</tr>
</tbody>
</table>

Theorem 29. We can see how each value expresses the electrical behavior of a very simple network. For example, the network with oriented matroid $\mathcal{N}_0(p_1) \oplus \mathcal{N}_1(p_2)$ constrains its port $p_1$, a loop to have voltage drop $v_1 = 0$ but its port $p_2$, a coloop to have current $i_2 = 0$. The current in the loop and voltage across the coloop are unconstrained. The subspace of 1-forms corresponding to equations ($v_1 = 0$; $i_2 = 0$) has basis $\{v_1, i_2\}$ and is therefore represented by any extensor $\alpha v_1 i_2$, $\alpha \neq 0$.

Similarly, the network of two parallel ports (case $\mathcal{N}_1^+$) constrains the sum of voltage drops going around the circuit to be 0, so Kirchhoff’s voltage law is expressed by $v_1 - v_2 = 0$. Kirchhoff’s current law in the same network is expressed $i_1 + i_2 = 0$. Hence the corresponding extensor is $\pm(v_1 - v_2)(i_1 + i_2)$.

Extensor $M_E(\mathbb{N})$ of Definition 14 is a representation of the linear subspace of constraints whose solution set $M^\perp \subseteq \mathbb{R}^{P_I \cup P_V}$ comprises all feasible combinations of voltage and current values at the network ports. When $M_E(\mathbb{N})$ is written as the exterior product of step 1 extensors, the factors are easily written as the rows of a matrix whose columns are indexed by $P_I \cup P_V$. Then $M^\perp = \{m \mid M_E(\mathbb{N}) m = 0\}$. Continuing our example:

<table>
<thead>
<tr>
<th>$\mathcal{N}$</th>
<th>(matrix)$M_0(\mathcal{N})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N}_0(p_1) \oplus \mathcal{N}_1(p_2)$</td>
<td>$\begin{bmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\mathcal{N}_1^+$</td>
<td>$\begin{bmatrix} 1 &amp; -1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
**Derivation of Theorem 47 and its Corollary:** Let \( M_E(N) \) be the extensor from Definition 14 for the two port network of Theorem 47. Applying Proposition 19 with all \( r_e = 1 \), our hypothesis about \( T \) implies \( \epsilon(p_1p_2)E\{N[v_1v_2] = \sum g_r > 0 \). Hence the equations \( M_E(N)(10 \rho_{11} \rho_{21})^t = 0 \) have a unique solution. From Cramer’s rule

\[
\rho_{11} = -\frac{M_E(N)[i_1v_2]}{M_E(N)[v_1v_2]} \quad \rho_{21} = -\frac{M_E(N)[v_1i_1]}{M_E(N)[v_1v_2]}
\]

We complete the derivation from Corollary 31. First for the denominators. We see from figure 2 that the only terms in (20) that might contribute to \( M_E(N)[v_1v_2] \) are those for which \( \mathcal{N}/A|_P \) is the matroid of two loops \( \mathcal{N}_0(p_1) \oplus \mathcal{N}_0(p_2) \) because the only appearance of \( v_1v_2 \) is in that matroid’s row. The rank conditions further restrict the contributing \( A \) to spanning trees of \( \mathcal{N}\{p_1\} \).

Consider the numerator \( M_E(N)[i_1v_2] \). Extensor \( \pm i_1v_2 \) is a term in \( M_0(\ldots) \) for three of the oriented matroids in figure 2. Note that all three appearances have the same coefficient +1. Hence all the \( F \) that contribute to this numerator contribute with the same sign. The rank conditions again determine that the contributing sets \( F \) are the bases in \( \mathcal{N}/p_1\setminus p_2 \).

Finally, for the numerator \( M_E(N)[v_1i_1] \), we locate \( \pm v_1i_1 \) in just the bottom two rows in figure 2. These appearances have opposite sign. For \( A \subseteq E \) with \( \mathcal{N}/A|_P = \mathcal{N}_1^- \), the contribution is \( -g_A \). The corresponding sign in (29) is + because of the sign change in Cramer’s rule. The sign is opposite when \( \mathcal{N}/A|_P = \mathcal{N}_1^+ \), so the distinct orientations of the 2-circuit obtained when contracting \( A \) account for the opposite signs in (29). We can again verify from the rank conditions that the \( F \) contributing to the numerator of (29) are the spanning forests with 2 trees containing the indicted vertices as claimed.

We remark that the sign dependancies of \( M_E(N) \) on \( \epsilon \) and the order of \( P = p_1p_2 \) cancel in the ratios for \( \rho_{11} \) and \( \rho_{21} \).

While Theorem 47 can be proved by elementary arguments as in [Che76], the above proof demonstrates how it can be derived from the forgoing theory using algebraic calculations.

### 8 Concluding Discussions

It is natural to generalize equations (7) so \( f_I(C_I) \) and \( f_v(C_V) \) are not orthogonal[Cha83]. Developing this along lines sketched in this paper leads to the directed graph version of the Matrix Tree Theorem. It did lead as well to a “oriented matroid pair” model for combinatorial conditions for certain equations with monotone non-linearities to be uniquely solvable[Cha96]. These conditions were stated in terms of two oriented matroids with a common ground set having complementary rank and no common non-zero covector; the current paper provides the insight that these two were obtained by deletion/contractions to eliminate port elements. Investigations of a generalization of the Tutte polynomial to two matroids with a common ground set were also begun in [WK04].

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**Paper in preparation (2004), #R00** 38
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